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Neuro-Psychological Interpretation of Mathematical Results Reported in Case of Continuous-Time Hopfield Neural Networks

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ABSTRACT

In this paper, for the mathematical description of electrical phenomena (voltage state) appearing in nervous system, continuous-time Hopfield neural network is used. The equilibriums of the continuous-time Hopfield neural network are interpreted as equilibriums of the nervous system. An equilibrium for which the steady state is locally exponentially stable is interpreted as robust equilibrium of the nervous system. That is because a small perturbation of steady voltage, the network recover the steady voltage. A path of equilibrium states which steady states are locally exponentially stable is interpreted as a path of robust equilibriums of the nervous system. This is a way to follow in healthcare for conduct the nervous system from a pathologic robust equilibrium into in a non-pathologic robust equilibrium. For illustration, path of robust equilibriums to follow in nervous control is computed.

Keywords: Continuous-Time Hopfield Neural Network; Nervous System; Robust Equilibrium; Fragile Equilibrium, Repulsive Equilibrium, Nervous System Control

MSC: 37B25; 62M45; 65P20; 92B20

Introduction

Continuous-time Hopfield neural networks claims to be mathematical descriptions of electrical phenomena appearing in nervous system. If such neural network is used to describe associative memories, several locally exponentially stable equilibriums are desired for one external electrical input, as they store information and constitute distributed and parallel neural memory networks. In this case, the purpose of the mathematical model is the description of the locally exponentially stable steady states (existence, number, regions of attraction, bifurcation) so as to ensure the recall capability. Mathematical results on the existence, number, regions of attraction, estimation of the local convergence rate, and bifurcation in the case of continuous-time Hopfield neural networks are given in Balint, et al. [1]. Our aim here is to present neuro-psychological interpretations of some reported mathematical results presented in Balint, et al. [1] and in the papers referred herein. Namely we want to answer the following questions: what represent an equilibrium of the neural network for the nervous system? which kind of equilibriums exists? when a nervous system has one or several equilibrium for the same external electrical input? what means for a nervous system the transfer of a pathologic equilibrium into a non-pathologic equilibrium? what is the importance of the local exponential stability of a steady state in case of transfer?

what represent for a nervous system a path of locally exponentially steady states? The answer to these questions can be important for the computation of path of equilibriums having locally exponentially stable steady states for use them in healthcare.

Continuous-time Hopfield Neural Networks

According to Balint, et al. [1] formula (5.1) a continuous-time Hopfield-type neural network, describing the voltage evolution in a network of biological neurons, is a system of nonlinear differential equations of the form

$$\dot{x} = -a_i \times \sum_{j=1}^{j=n} T_{ij} \times g_j\left(x_j\right) + I_i \quad \text{i=1…n} \quad (2.1)$$

where: $a_i > 0$, I_i are constants: a_i related to the neuron i membrane capacity and I_i related to the external electrical input, $T = \left(T_{ij}\right)_{n\times n}$ is a constant matrix referred to as the interconnection matrix, $g_j : R \to R$, j=1…n represent the neuron input-output activations. The activation functions are bounded and without restraining generality, we may suppose that $\left|g_j(s)\right| \le 1$ for any $s \in R$, j=1…n . If it not mentioned otherwise, it is assumed that $g_j(0) = 0$, for j=1…n . The activation functions are increasing and have bounded derivatives. More precisely, there exist $k_j > 0$ such that $0 < g_j^{'}(s) \le k_j$ for any $s \in R$, j=1…n . Frequently the activation functions are assumed to verify $g_j(s) = 1$ if $s \ge 1$ and $g_j(s) = -1$ if $s \le -1$.

The system (2.1) can be written in matrix form:

$$\dot{X} = A \times X + T \times G(X) + I \quad (2.2)$$

Where $X = \left(x_1, x_2, \dots, x_n\right)^T$, $A = \text{diag}\left(-a_1, -a_2, \dots -a_n\right) \in M_{n\times n}$, $I = \left(I_1, I_2 \dots I_n\right)^T \in R^n$ and $G : R^n \to R^n$ is given by $G(X) = \left(g_1(x), g_2(x), \dots g_n(x)\right)^T$.

Let be $F : R^n \times R^n \to R^n$ the function given by $F(X, I) = A \times X + T \times G(X) + I$. With this function equation (2.2) can be written in the form:

$$\dot{X} = F(X, I) \quad (2.3)$$

By definition an equilibrium of (2.3) is a solution of the equation:

$$F(X, I) = 0 \quad (2.4)$$

In other words an equilibrium E is a couple (X, I) from $R^n \times R^n$ which verifies (2.4). An equilibrium $E = (X, I)$ (if exist) then I is called external electrical input and X is called steady state .The name steady state is justified by the fact that if $E^0 = \left(X^0, I^0\right)$ is an equilibrium then for I equal to I^0 the solution of (2.3) which verifies the initial condition $X(t_0) = X^0$ is constant equal to X^0. According to Balint, et al. [1]. for any given voltage steady state X taking the external electrical input given by the formula :

$$I = -A \times X - T \times G(X) \quad (2.5)$$

an equilibrium $E = (X, I)$ is obtained.

On the other hand theoretically it can happen that for a given input I^0 there is no, there exist one, or there exits several different voltage states X^j, $J = 1 \dots m$ such that $E^j = \left(X^j, I^0\right)$ for $J = 1 \dots m$ are equilibriums for (2.3).

Neuro-Psychological Interpretation of the Equilibrium

A natural neuro-psychological interpretation of the concept of equilibrium $E^0 = \left(X^0, I^0\right)$ of the neural network is that it represent an equilibrium of the nervous system. Hence, come the idea that in order to change a (none desired) pathological equilibrium $E^0 = \left(X^0, I^0\right)$ of the nervous system, a new external electrical input I^1 has to be applied. If the voltage component of the new non pathologic equilibrium is X^1 then it is natural to think that the new external electrical input I^1 ,which has to be applied, has to be taken according to the formula (2.5) i.e. $I^1 = -A \times X^1 - T \times G\left(X^1\right)$ hoping that after the change $I^0 \to I^1$ of the external electrical input , the voltage X^0 of the nervous system evolve to the voltage X^1 of the nervous system. Mathematically this neuro-psychological though is correct if after the moment t_1 when the change $I^0 \to I^1$ takes place the voltage state of the neural network, described by the solution of the initial value problem

$$\dot{X} = F\left(X, I^1\right) \quad , \quad X\left(t_1\right) = X^0 \quad (3.1)$$

tends to the voltage state X^1.

This kind of reasoning make sense if $I^1 \ne I^0$. That is because if $I^1 = I^0$ then there is no change in input and the voltage state of the neural network will rest in the state X^0 i.e. the voltage state evolution of the neural network is described by (3.1) is constant equal to X^0.

Moreover, even if $I^1 = I^0$ and the reasoning make sense, it can happen that for the new electrical input I^1,beside the desired voltage state X^1, there exist a second voltage state X^2, and applying the electrical input I^1 beside the non-pathological equilibrium $E^1 = \left(X^1, I^1\right)$ a second equilibrium $E^2 = \left(X^2, I^1\right)$ appear. It can happen that the equilibrium $\left(X^2, I^1\right)$ is pathologic too. Therefore the problem is to find supplementary condition assuring that the solution of the initial value problem (3.1) tends to X^1 as it was planned.

In Balint, et al. [1] provide computational simulation of the above-described phenomena.

The neural network (5.2) considered in Balint, et al. [1] is defined by the system of differential equations:

$$\dot{x}_1 = -x_1 + \frac{17 \times ln4}{15} \times \tanh x_2 + I_1 \quad , \quad \dot{x}_2 = -x_2 + \frac{17 \times ln4}{15} \times \tanh x_1 + I_2 \quad (3.2)$$

For obtain the prior given steady voltage state $X = (1,2)^T$ according to (2.5) the external input which has to be applied is $I = \left(-0.514616132, 0.803433824\right)^T$. In order to see the voltage evolution of the neural network the following initial value problem has to be solved:

$$\dot{x}_1 = -x_1 + \frac{17\times ln4}{15}\times \tanh x_2 - 0.514616132$$

$$x_1(0)=1 \quad \dot{x}_2 = -x_2 + \frac{17\times ln4}{15}\times \tanh x_1 + 0.803433824$$

$$x_2(0)=2 \qquad (3.3)$$

The solution of the initial value problem (3.3) is represented on the Figures 1 & 2. These figures show that maintaining the external input value in the first five seconds the voltage of the neural network is constant equal to the initial value and after that oscillate around the initial value with an amplitude less than 10-6. This means that practically there is no change in the voltage of the neural network. According to the neuro-psychological interpretation, this type of the neural network voltage behavior indicates that $EE=(X,I)$ is an equilibrium of the corresponding nervous system. If the steady state $E=(X,I)$ is pathologic then a neurological or psychological intervention is needed. The change of the external electrical input represents a type of intervention. Assume that the medical decision is to transform the pathologic equilibrium $E=(X,I)$ with $X=(1,2)^T$ and $I=(-0.514616132, 0.803433824)^T$ into the new non-pathologic equilibrium $E^1=(X^0, I^1)$ with $X^0=(0,0)^T$ and $I^1=(0,0)^T$ computed using formula (2.5). Before describing the effect of the external input change $I\rightarrow I^1$ remark that for the new external input $I^1=(0,0)^T$ the system (3.2) possesses several equilibriums: $E^1=((0,0)^T,(0,0)^T)$, $E^2=((ln4, ln4),^T (0,0)^T)$, $E^3=((-ln4,-ln4),^T (0,0)^T)$. In case of the above equilibriums the steady states can be obtained solving the system of nonlinear algebraic equations:

$$-x_1 + \frac{17\times ln4}{15}\times \tanh x_2 = 0, \quad -x_2 + \frac{17\times ln4}{15}\times \tanh x_1 = 0 \qquad (3.4)$$

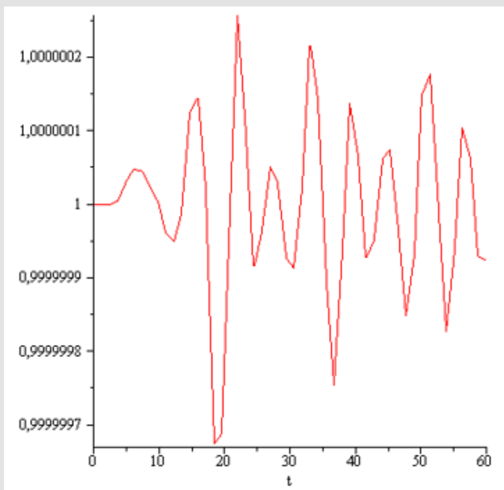

Figure 1: x_1 versus t.

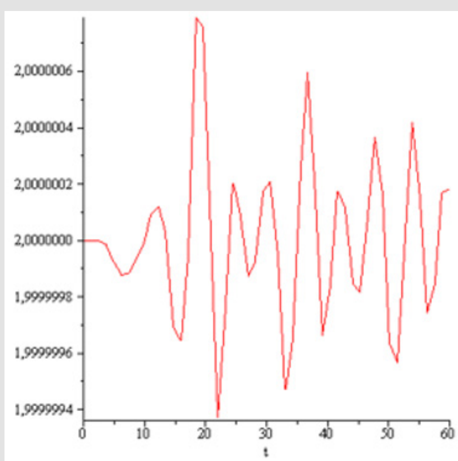

Figure 2: x_2 versus t.

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The effect of the external input change $I \to I^1$ can be seen solving the following initial value problem:

$$\dot{x}_1 = -x_1 + \frac{17 \times ln4}{15} \times \tanh x_2 \quad x_1(0) = 1$$

,

$$\dot{x}_2 = -x_2 + \frac{17 \times ln4}{15} \times \tanh x_1 \quad x_2(0) = 2 \quad (3.5)$$

The solution of the initial value problem (3.5) is represented on the figures (Figure 3 & 4).

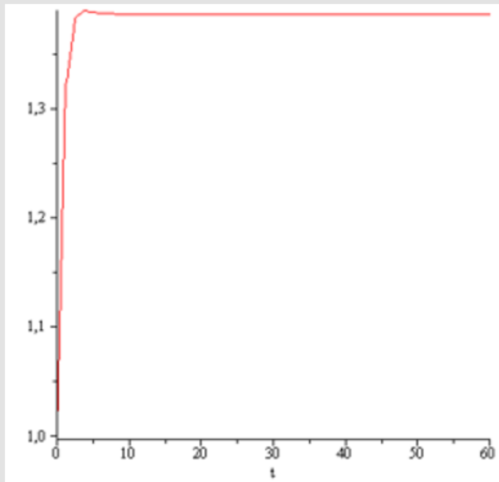

Figure 3: x_1 versus t.

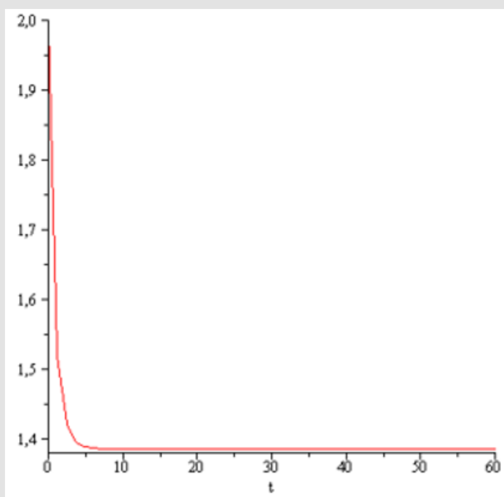

Figure 4: x_2 versus t.

These figures show that the effect of the change $I \to I^1$ is the transfer of the steady state $(1,2)^T$ into the steady state $(ln4, ln4),^T$ and not into the steady state $(0,0)^T$ as were expected.

The mathematical explanation is: the steady state $(ln4, ln4),^T$ is locally exponentially stable and the steady state $(1,2)^T$ belongs to the region of attraction of the steady state $(ln4, ln4),^T$. In the same time, the steady state ⟦(0,0)⟧^T is unstable and repulsive. The unstable character of steady state $(0,0)^T$ means that for any small perturbation of initial condition, the solution of the perturbed initial value problem

$$\dot{x}_1 = -x_1 + \frac{17 \times ln4}{15} \times \tanh x_2 \quad x_1(0) = \varepsilon,$$

$$\dot{x}_2 = -x_2 + \frac{17 \times ln4}{15} \times \tanh x_1 \quad x_2(0) = \delta \quad (3.6)$$

do not recover the steady state $(0,0)^T$.

The unstable character of the steady state voltage in case of equilibrium E^1 is illustrated on the next figures (Figures 5 & 6).

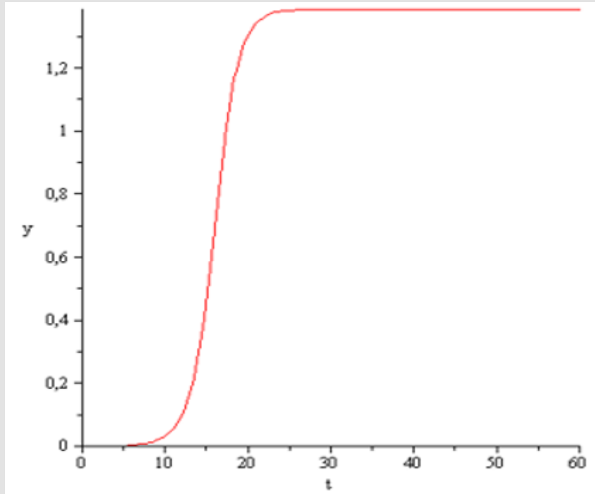

Figure 5: $x_1$versus t.

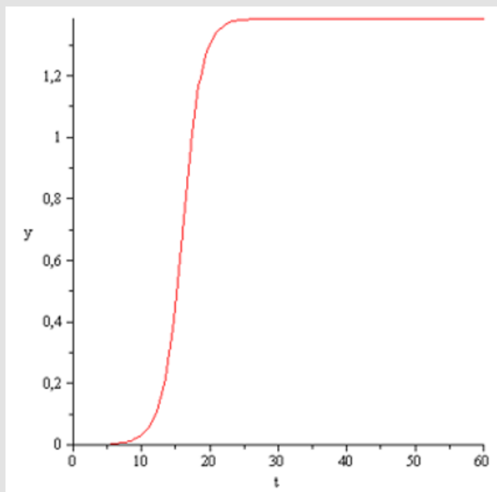

Figure 6: x_2 versus t.

The above figures illustrate also the repulsive character of the equilibrium $E^1=((0,0)^T,(0,0)^T)$. That is because the solution of the initial value problem (3.6) represent also the evolution of the steady state $X^{\varepsilon,\delta}=(\varepsilon,\delta)^T$ of an equilibrium $E^{\varepsilon,\delta}=(X^{\varepsilon,\delta},I^{\varepsilon,\delta})$ for $I^{\varepsilon,\delta}\to(0,0)^T$.The external electrical input $I^{\varepsilon,\delta}$ appearing here is obtained from the steady state $X^{\varepsilon,\delta}$ using formula (2.5). Figures shows that the transfer $E^{\varepsilon,\delta}\to E^1=(0,0)^T$ is not possible because the components of the steady state $X^{\varepsilon,\delta}$ move away from the steady state $X^1=(0,0)^T$ According to the neuro-psychological interpretation of equilibrium, is important to keep in mind that in a nervous system there are three types of equilibriums: -Equilibriums of first type for which after a small perturbation of the steady state the nervous system return to the equilibrium. This is the situation if the steady state of the corresponding neural network is locally exponentially stable. (as is the equilibrium $E^2=E^2=((l\,\mathrm{n}\,4,l\,\mathrm{n}\,4),^T,I^1))$.Due to this property the nervous system return to the equilibrium automatically ,without any external input. We will say that this equilibrium of the nervous system is robust.

-Equilibriums of second type for which after a small perturbation of the steady state the nervous system do not return to the equilibrium. This is the situation if the steady state of the corresponding neural network is unstable. (as is the equilibrium $E=(X,I)$ with $\mathrm{I}=(1,2)^T$ and $I=(-0.514616132,0.803433824)^T$. Due to the property that the nervous system do not return to the equilibrium automatically, without applying an external input, we will say that this type of equilibrium of the nervous system is fragile. --Equilibriums of third type having the property that there is no equilibrium, which can be transferred in such type of equilibrium. Due to this property, we will say that this type of equilibrium of the nervous system is repulsive.

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Equilibriums Transfer

A correct neuro psychological interpretation and understanding of the possible equilibriums of the neural network permit to neurologist and psychologist to choose appropriate tool in a specific case. On this basis people, working in neural and mental healthcare, can choose appropriate tool for transfer the pathologic equilibrium of a patient into a non-pathologic equilibrium. The choice of the appropriate tool means : start from a fragile or robust pathologic equilibrium $E^0=\left(X^0,I^0\right)$, fix a new non-pathologic steady state X^1 and compute for X^1, the corresponding new external electrical input I^1 (using formula (2.5)), build up the new non-pathologic equilibrium $E^1=\left(X^1,I^1\right)$ .After that, several computations has to be made in order to be able to say that step by step the transfer $E^0=\left(X^0,I^0\right)\to E^1=\left(X^1,I^1\right)$ is possible.

1. Verify if the equilibrium E^0 is fragile or robust. A way is to solve and represent the solutions of the initial value problems

$$\dot{X}=F\left(X,I^0\right)\quad X\left(0\right)=X_1^0 \quad (4.1)$$

where X_1^0 are small perturbations of X^0.

2. Verify if the equilibrium E^1 is fragile or robust. A way is to solve and represent the solutions of the initial value problems

$$\dot{X}=F\left(X,I^1\right)\quad X\left(0\right)=X_1^1 \quad (4.2)$$

where X_1^1 are small perturbations of X^1.

3. Verify if the region of attraction of the steady state X^1 contains the steady state X^0. A way is to solve and represent the solution of the initial value problem:

$$\dot{X}=F\left(X,I^1\right)\quad X\left(0\right)=X^0 \quad (4.3)$$

4. Verify if the region of attraction of the steady state X^0 contains the steady state X^1. A way is to solve and represent the solution of the initial value problem:

$$\dot{X}=F\left(X,I^0\right)\quad X\left(0\right)=X^1 \quad (4.4.)$$

In order to see how this work in practice, consider the neural network (3.2) and the equilibrium $E^0=\left(X^0,I^0\right)=\left((1,1)^T,(-0.196566176,-0.196566176)^T\right)$. Fix the new steady state $X^1=(1.1,1.1)^T$ and using (2.5) compute the corresponding new external electrical input I^1 finding $I^1=\left(-0.157690917,-0.157690917\right)^T$. So the new equilibrium is $E^1=\left(X^1,I^1\right)$.

-For test the fragility or robustness of $E^0=\left(X^0,I^0\right)$, solve and represent the initial value problem (4.1). Taking for example $X_1^0=\left(1.11,1.11\right)$. The solution of the initial value problem (4.1) is presented on the next figures (Figures 7 & 8):

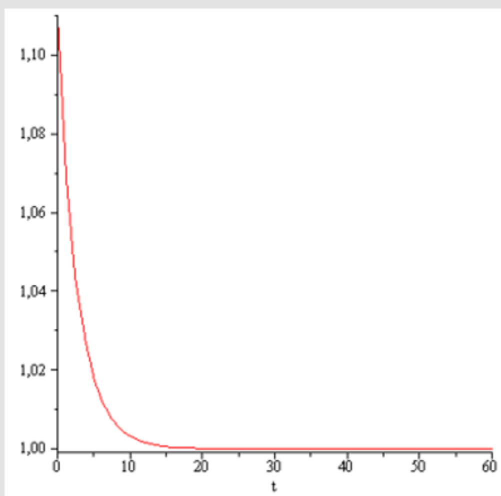

Figure 7: $x_1$versus t.

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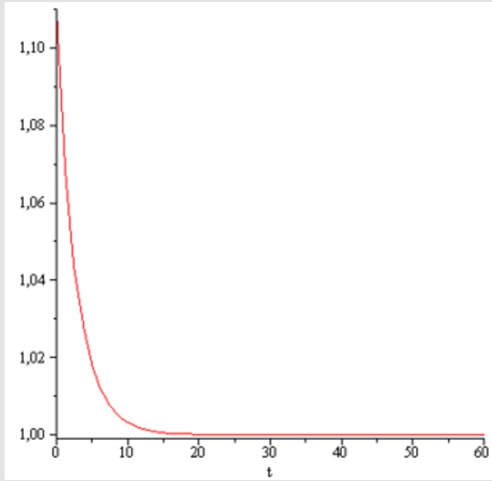

Figure 8: x_2 versus t.

These figures suggest that the equilibrium E^0 is robust.

-For test the fragility or robustness of $E^1 = (X^1, I^1)$, solve and represent the initial value problem (4.2).

Taking for example $X_1^1 = (1.01, 1.01)$ the solution of the initial value problem (4.2) is presented on the next figures (Figures 9 & 10).

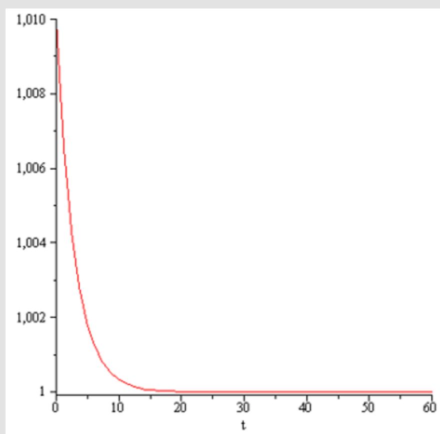

Figure 9: x_1 versus t.

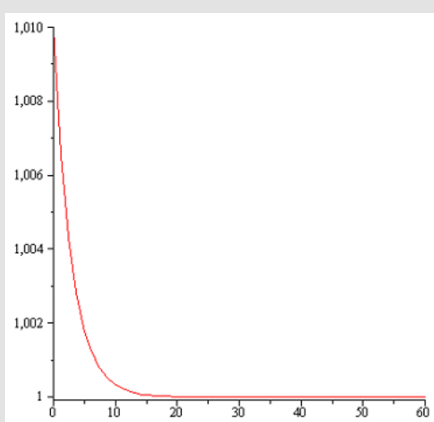

Figure 10: x_2 versus t.

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These figures suggest that the equilibrium E^1 is robust.

-For test if the change of the external electrical input $I^0 \rightarrow I^1$ lead to the transfer $E^0 \rightarrow E^1$ the initial value problem (4.3) has to be solved.

The solution of the initial value problem (4.3) is represented on the next figures (Figures 11 & 12).

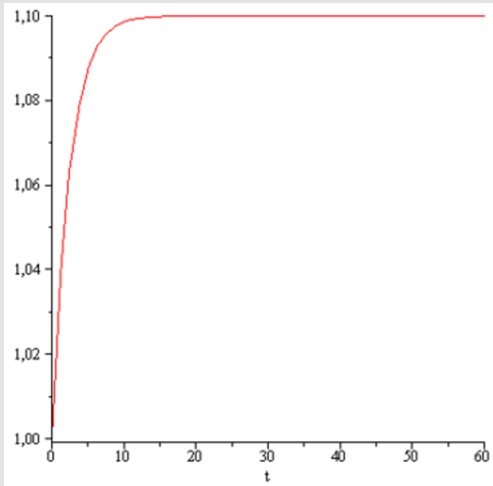

Figure 11: $x_1$versus t.

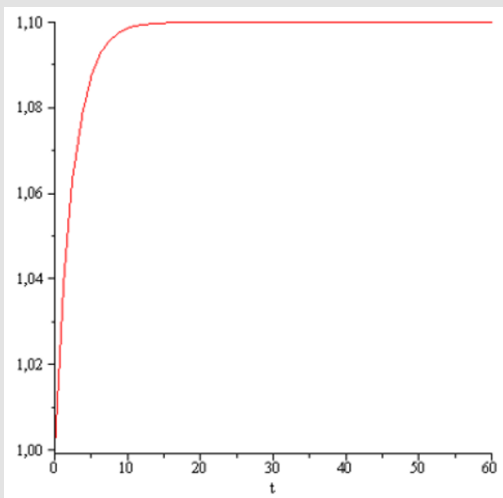

Figure 12: x_2 versus t.

These figures suggest that the change of the external electrical input $I^0 \rightarrow I^1$ lead to the transfer $E^0 \rightarrow E^1$.

For test if the change of the external electrical input $I^1 \rightarrow I^0$ lead to the transfer $E^1 \rightarrow E^0$ the initial value problem (4.4) has to be solved.

The solution of the initial value problem (4.4) is represented on the next figures (Figures 13 & 14).

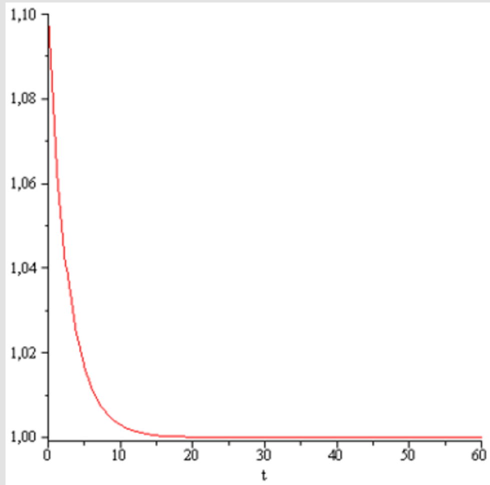

Figure 13: x_1 versus t.

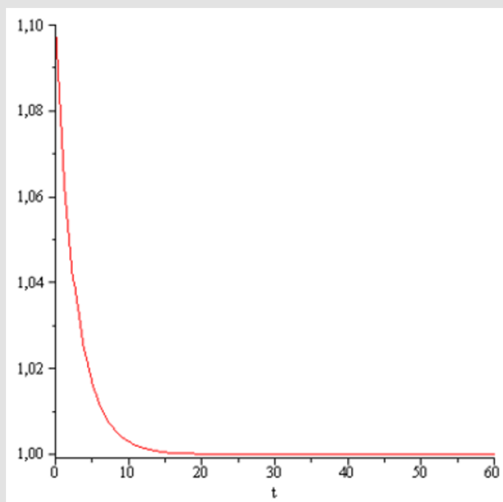

Figure 14: x_2 versus t.

These figures suggest that the change of the external electrical input $I^1 \to I^0$ lead to the transfer $E^1 \to E^0$.

Remember that the starting motivation of the above presented computations was that the equilibrium E^0 is pathologic and the equilibrium E^1 is non-pathologic and an external medical intervention is necessary. We underline that in general the situation is much more complex concerning the configuration of the equilibriums, the type of equilibrium (robust, fragile, and repulsive) and the transfer of an equilibrium into another equilibrium. In the following, we present results from Balint, et al. [1], which can offer an overview about the complexity. Illustrative computational examples are given. In Balint, et al. [1] states: If Δ is a rectangle in R^n,(i.e. for i=1…n there exist α_i, $\beta_i \in R$, $\alpha_i < \beta_i$, such that $\Delta = (\alpha_1,\beta_1)\times(\alpha_2,\beta_2)\times\ldots\times(\alpha_n,\beta_n))$ and $\det(A + T \times DG(X)) \neq 0$ for any $X \in \Delta$,then the function I_Δ ,the restriction of the external electrical input function $I(X) = -A \times X - T \times G(X)$, is injective. This theorem reveal that in a prior given rectangle Δ (included in R^n) if $\det(A + T \times DG(X)) \neq 0$ for any $X \in \Delta$, then for any $X^0 \in \Delta$ the input $I(X^0) = -A \times X^0 - T \times G(X^0)$ is unique. Therefore, the equilibrium $E^0 = (X^0, I^0)$ of the nervous system is unique. This situation is completely different from that described in Example (5.2) pg.184 [1] where in case of the rectangle $\Delta = (-1.5,1.5)\times(-1.5,1.5)$ for the input $I^0 = (0,0)^T$ three different equilibriums $E_1^0 = ((0,0)^T, I^0)$, $E_2^0 = ((\ln 4, \ln 4)^T, I^0)$ and $E_3^0 = ((-\ln 4, -\ln 4)^T, I^0)$ exists each of them having the steady state in Δ. A numerical illustration of the situation described by Theorem 5.3. pg. 175[1]. can be found in Example 5.1. pg. 183 [1]. In this example the following one- dimensional neural continuous-time Hopfield neural network is considered:

$$\dot{x} = -a \times x \_ T \times \tanh x + I \quad (4.5)$$

where $a > 0, T$ *and* I are constants.

For any state $x \in \Delta = (-\infty, \infty)$ the external input $I(x)$ for which x

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is a steady state of (4.5) is

$$I(x) = a \times x - T \times \tanh x.a$$

-If $T < a$ then $I'(x) > 0$ *for any* $x \in \Delta, x \neq 0$ Therefore for any $x^0 \in \Delta = (-\infty, \infty)$ the external input $I(x^0) = a \times X^0 - T \times \tanh x^0$ is unique. -If $T = a$ then $I'(x) > 0$ *foranyx* $\in \Delta, x \neq 0$ and

$x^0 \in \Delta_1 = (-\infty, 0)$ *the external inputI*$(x^0) = a \times x^0 - T \times \tanh x^0$ *is unique*,

and for any

$x^0 \in \Delta_2 = (-0, \infty)$ *the external inputI*$(x^0) = a \times X^0 - T \times \tanh x^0$ *is unique*

-If $T > a$ then $I'(x) \neq 0$ *for any* $x \in \Delta, x \neq x^1 = -\arctan h\sqrt{\frac{T-a}{T}}$ *and* $x \neq x^2 \arctan h\sqrt{\frac{T-a}{T}}$ and $I'(x^1) = I'(x^2) = 0$ Therefore for any

$x^0 \in \Delta_1 = (-\infty, x^1)$ *the external input* $I(x^0) = a \times x^0 - T \times \tanh x^0$*is unique*,

for any

$x^0 \in \Delta_2 = (x^1, x^2)$ *the external input* $I(x^0) = a \times x^0 - T \times \tanh x^0$*is unique*,

and for any

$x^0 \in \Delta_3 = (x^2, \infty)$ *the external input* $I(x^0) = a \times X^0 - T \times \tanh x^0$*is unique*.

In Balint and..2008 theorem 5.4. pg. 176 states: For any prior given external electrical input $X \in R^n$ nthe following, statements hold: -There exists at least one steady state $X \in R^n$ in the rectangle $\Delta = [-M_1, M_1] \times [-M_2, M_2] \times \ldots \times [-M_n, M_n] \subset R^n$, where $M_i = \frac{1}{a_i}(|I_i| + \Sigma_{j=1}^n |T_{ij}|)$ for i=1…n, such that $F(X, I) = 0$. -Every steady state X, solution of the equation $F(X, I) = 0$,belongs to the rectangle Δ defined above. -If in addition $\det(A + T \times DG(X)) \neq 0$ for any $X \in \Delta$, then the equation $F(X, I) = 0$ has a unique solution $X \in \Delta$.

This theorem clarify several things: -First, the theorem assure that applying an arbitrary prior given external electrical input $I^0 \in R^n$ to the nervous system there exist at least one steady state X^0 such that $E^0 = (X^0, I^0)$ is an equilibrium of the nervous system. The steady state X^0 of the equilibrium $E^0 = (X^0, I^0)$ is located in the rectangle Δ specified above. This is in fact mainly a localization of the equilibrium steady state. For find effective, the steady state X^0, the nonlinear algebraic equation $F(X, I^0) = 0$ has to be solved in Δ .In case of the nervous system described by the one- dimensional neural network (4.5) for $T = a = 1$ and $I = 1$ the rectangle $\Delta = [-2, 2]$ .For the nonlinear algebraic equation $-x + \tanh x + 1 = 0$ a solution has to be searched in the rectangle $\Delta = [-2, 2]$.By solving this equation the solution $x^0 = 1.961179751$ is found. -Second, the theorem assure that applying an arbitrary prior given external electrical input $I^0 \in R^n$ to the nervous system, every steady state X^0which appear due to that is in the rectangle Δ . -Third if $\det(A + T \times DG(X)) \neq 0$ for any $X \in \Delta$, (specified above) then the obtained steady state X^0 is unique. This means that the obtained equilibrium $E^0 = (X^0, I^0)$ is unique. In case of the nervous system described by the one- dimensional neural network (4.5) for $T = 1, a = 2, I = 1$ and the rectangle $\Delta = [-1, 1]$, $I^1(x) > 0$ *for any* $x \in \Delta$ .Therefore the nonlinear algebraic equation $-2 \times x + \tanh x + 1 = 0$ has a unique solution in $\Delta = [-1, 1]$. By solving this equaton, the solution $X^0 = 0.8439469994$ is found. The supplementary information is that X^0 is unique and the equilibrium $E^0 = (X^0, I^0)$ is unique. In Balint, et al. [1] states that if the neuron input-output activation functions verifies the general conditions described in section2 and for an external input I the following inequalities hold

$$|I_i| < T_{ii} - a_i - \Sigma_{i \neq j} |T_{ij}| \quad i = 1, 2, \ldots n \quad (4.6)$$

then in every rectangle Δ_ε $\varepsilon \in \{\pm 1\}$ there exists a unique steady state $X^{\varepsilon, I}$ such that $E^{\varepsilon, I} = (X^{\varepsilon, I}, I)$ is an equilibrium of the neural network (2.3). Here

$$\Delta_\varepsilon = J(\varepsilon_1) \times J(\varepsilon_2) \times \ldots J(\varepsilon_n),$$

$$J(-1) = (-\infty, -1), J(1) = (1, \infty).$$

The mathematical condition (4.6) concerns the magnitude of the external input (left hand side) and the coefficients of the neural network (right hand side). If an input I^0, which verifies (4.6), is applied to the nervous system then due to that, in the nervous system, 2^n *equilibriums*$E^{\varepsilon, I^0} = (X^{\varepsilon, I^0}, I^0)$*appear* .Each steady state X^{ε, I^0} is unique and located in a rectangle Δ_ε. This is an extremely complex configuration of steady states, which appear after applying an external electrical input I^0. A modified variant of the above theorem, is Theorem (5.7) pg.178 Balint, et al. [1]. According to Theorem 5.7. pg. 178. Balint and..2008, the next statement hold. If there exits $\alpha \in (0,1)$ such that the functions g_i, $i = 1, 2, \ldots n$ satisfy:

$$g_i(s) \geq \alpha \text{ if } s \geq 1 \quad \text{and} \quad g_i(s) \leq \alpha \text{ if } s \leq -1 \quad \text{for, } i = 1, 2, \ldots n \quad (4.7)$$

and in addition, the external input $I \in R^n$ satisfies

$$|I_i| < T_{ii} \times a - a_i - \Sigma_{i \neq j} |T_{ij}| \quad \text{for, } i = 1, 2, \ldots n \quad (4.8)$$

then the following conclusions hold:

i). In every rectangle $\Delta_\varepsilon, \varepsilon \in \{\pm 1\}$ there exists at least one steady state

$X^{\varepsilon, I}$ *such that* $E^{\varepsilon, I} = (X^{\varepsilon, I}, I)$*is an equilibrium of the neural network*(2.3).

ii). Every rectangle $\overline{\Delta_\varepsilon}$, is invariant to the voltage dynamic of the network. This theorem reveal that if the neuron input-output activations verify (4.7) and one input I^0,which verifies (4.8), is applied to the nervous system then due to that in the nervous system

$$2^n \textit{ equilibriums} E^{\varepsilon, I^0} = (X^{\varepsilon, I^0}, I^0) \textit{appear}.$$

Each steady state X^{ε, I^0} is unique and located in a rectangle Δ_ε. This configuration of steady states, is similar which appear in theorem 5.6. What is new is the invariance of $\overline{\Delta_\varepsilon}$ to the voltage dynamics. According to Theorem 5.10. pg. 179. Balint, et al. [1] if the conditions of Theorem 5.6. pg. 177 are satisfied then the steady state $X^{\varepsilon, I}$ corresponding to I and belonging to Δ_ε is locally exponentially stable, and its region of attraction includes $\overline{\Delta_\varepsilon}$. A numerical illustration of the phenomena described in Theorem 5.7. pg. 178. Balint, et al. [1] is given in Example (5.3) pg.186 Balint, et al. [1]. In this example the following Hopfield neural network is considered:

$$\dot{x}_1 = -a_1 \times x_1 + b_1 \times g(x_1) + b_2 \times g(x_2) + I_1$$

$$\dot{x}_2 = -a_2 \times x_2 + b_2 \times g(x_1) + b_1 \times g(x_2) + I_2$$

(4.9)

where: $g: R \to (-1,1)$, $g(s) = \frac{2}{\pi} \times \arctan(\frac{\pi}{2} \times s)$.Let be $\alpha = g(1) \approx 0.63$ and $\beta = g(1) \approx 0.28$. One can check that $g(s) \geq \alpha\ if\ s \geq 1$ and $(s) \leq -\alpha\ if\ s \leq -1$; moreover $0 < g(s) \leq \beta$ for any $|s| \geq 1$.

It can be proven that if

$$\beta \times (|b_1| + |b_2|) < a_i < a \times b_1 - |b_2| \text{ for } i = 1,2 \quad (4.10)$$

then for any input $I = (I_1, I_2)$ satisfying

$$|I_i| < a \times b_1 - |b_2| - a_i \quad \text{for} \quad i = 1,2 \quad (4.11)$$

there exists a unique steady state $X^{\varepsilon,I}$ in every rectangle Δ_ε. It is locally exponentially stable and its region of attraction includes $\overline{\Delta_\varepsilon}$. For $b_1 = 1000$, $b_1 = -0.5$,and $a_1 = a_2 = \alpha \times b_1 - |b_2| - 300 = -19.860$ =-19.860 it follows that for any input I such that $|I_i| < 300 \quad i = 1,2$ in every rectangle Δ_ε there exists a unique steady state $X^{\varepsilon,I}$ which is locally exponentially stable and whose region of attraction includes $\overline{\Delta_\varepsilon}$.The four rectangles Δ_ε are:

$$\Delta_{1,1} = (1,\infty) \times (1,\infty), \Delta_{-1,1} = (-1,-\infty) \times (1,\infty),$$

$$\Delta_{-1,-1} = (-1,-\infty) \times (-1,-\infty), \Delta_{1,-1} = (1,\infty) \times (-1,-\infty)$$

(4.12)

Let be $S_\varepsilon = \{\frac{x_i^{\varepsilon I}}{|I_i|} < 300, i = 1,2\} \subset \Delta_\varepsilon$.In the next figure the gray rectangles represent the four sets S_ε The four spirals in Figure 15. represents locally exponentially stable steady states corresponding to the inputs $I^u = (20 \times u \times \cos u, 20 \times u \times \sin u)\ with\ u \in [0, 2\pi]$ .Each spiral is a path of robust equilibriums of nervous system. In other words, each spiral is a possible way to follow in healthcare in order to transfer pathologic equilibriums in non-pathologic equilibrium. (Figure 15). Path of robust equilibriums. In the following a numerical illustration is given. For

$$u = \frac{\pi}{6}, \quad X^{\frac{\pi}{6}} = (2.599706777, -2.547086602)^T$$

$$, \quad I^{\frac{\pi}{6}} = (9.068996827, 5.235987758)^T \text{ for}$$

$$u = \frac{\pi}{4}, \quad X^{\frac{\pi}{4}} = (2.607168246, -2.525360398)^T,$$

$$I^{\frac{\pi}{4}} = (11.10720734, 10720734)^T.$$

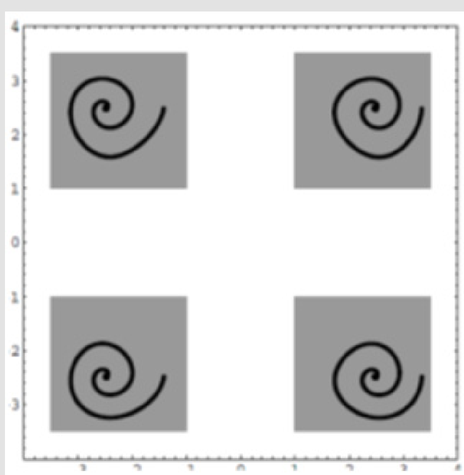

Figure 15: Path of robust equilibriums.

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For seeing that the steady state $X^{\frac{\pi}{6}}$ can be transferred into the steady state $X^{\frac{\pi}{4}}$ by making the external input change $I^{\frac{\pi}{6}} \to I^{\frac{\pi}{4}}$ the following initial value problem has to be solved:

$$\dot{x}_1 = -a_1 \times x_1 + b_1 \times g(x_1) + b_2 \times g(x_2) + I_1^{\frac{\pi}{4}};$$

$$\dot{x}_2 = -a_2 \times x_2 + b_2 \times g(x_1) + b_1 \times g(x_2) + I_1^{\frac{\pi}{4}} \quad (4.13)$$

$$x_1(0) = 2.599706777, \quad x_2(0) = -2547086602.$$

The solution of (4.13) is represented in (Figures 7 & 8). (Figures 16 & 17) show that the external input change $I^{\frac{\pi}{6}} \to I^{\frac{\pi}{4}}$ lead the steady state $X^{\frac{\pi}{6}}$ into the steady state 3. This example intends to illustrates the existence of several robust equilibrium paths and the equilibrium steady states transfer along the equilibrium pats. The Artificial Intelligence strategy for healthcare has to be the buildup path of locally exponentially stable steady states along which by small successive changes, the neural network voltage can be conducted, through the regions of attraction of intermediary locally exponentially stable steady states to the final non pathologic steady state.

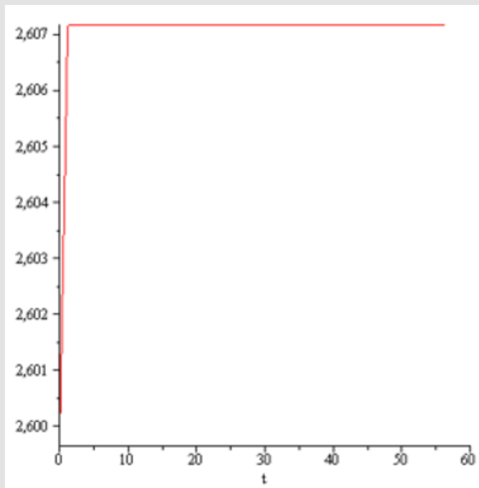

Figure 16: x_1 versus t.

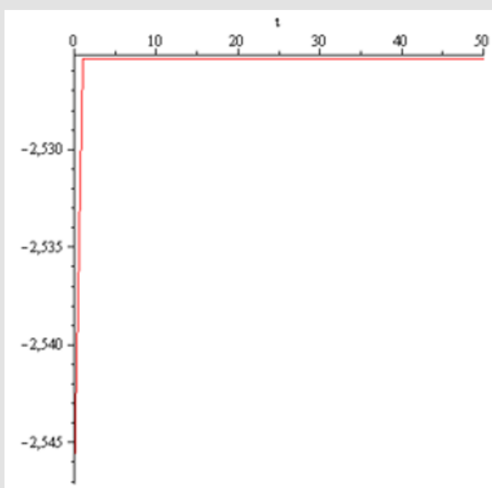

Figure 17: x_2 versus t.

Conclusion

The interpretation of mathematical results reported in Balint, et al. [1] reveal that the configuration of the possible equilibriums in nervous system is very complex. There exists different kind of equilibriums: fragile, robust and repulsive. In equilibrium, the voltage state of the neural network is constant and does not change if the external electrical input value is maintained constant. For this reason, if an equilibrium is pathologic then a neurological or psychological intervention is needed. The analysis presented in the paper reveal a way to follow in the practice for the treatment of a pathologic equilibrium. That is to connect the pathologic equilibrium with a non-pathologic equilibrium computing a path of robust equilibriums and make transfer gradually following the path. A treatment procedure usually follows such a path.

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