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An Optimal Robin-Type Domain Decomposition Method for Raviart-Thomas Vector Field in Three Dimensions

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ABSTRACT

In this paper, we present a Robin-type nonoverlapping domain decomposition (DD) preconditioner, which is deduced from the renew equation of the Robin-Robin iteration method. The unknown variables to be solved in this preconditioned algebraic system are the Robin transmission data on the interface. Through choosing suitable parameter on each subdomain boundary and using the tool of energy estimate, for Raviart-Thomas vector field in three dimensions, we prove that the condition number of the preconditioned system is O (1), and the DD method is optimal. I prove the discrete extension theorem in H(div). Numerical results are given to illustrate the efficiency of our DD preconditioner.

Keywords: Finite Element Method; Robin-Type Domain Decomposition Method; Condition Number Estimate; Raviart-Thomas Vector Field; Optimal

Introduction

In this paper, we consider the following boundary value problem

$$\begin{cases} -grad(adivu\_+Bu=fin\Omega \\ u.n=0in\Omega \end{cases} \quad (1.1)$$

Given a bounded convex polyhedral domain $\Omega\subset R^3$, we introduce the boundary value problems where $a\in L^\infty(\Omega)$ is a scalar-valued positive function bounded away from zero, the coefficient matrices $B+[b_{ij}]$ is symmetric uniformly positive definite with $b_{ij}\in L^\infty(\Omega), 1\le i,j\le n$, and $f\in L^2(\Omega)^3$. The weak formulation of problem (1.1), and the study of the Raviart-Thomas finite elements, as well as our iterative method, require the introduction of an appropriate Hilbert space $H(div;\Omega)$. It is given by

$$H(div;\Omega):=\{v\in L^2(\Omega)^3 \mid divv\in L^2(\Omega)\}$$

which equipped with the inner product $(.,.)div;\Omega$; and the associated graph norm $\|.\|div;\Omega$ defined

by

$$(u,v)div;\Omega:=\int_\Omega u.vdx+\int_\Omega divu.divvdx,\quad \|u\|^2_{div;\Omega}=(u,u)div;\Omega$$

The subspace of vectors in $H(div;\Omega)$ with vanishing normal component on $\partial\Omega$ is denoted by

$H_0(div;\Omega)$ and it is the appropriate space for the variational formulation of (1.1):

Find $u\in H_0(div;\Omega)$ such that

$$a(u.v)=\int_\Omega f.vdx,\quad v\in H_0H_0(div;\Omega),$$

where the bilinear form $a(.,.)$ is given by

$$a(u.v)=\int_\Omega (adivu.divv+Bu.v)dx,\ u,v\in H(div;\Omega).$$

We associate an energy norm, defined by $\|.\|_a^2:=a(.,.)$, with the bilinear form; our associations on the coefficients guarantee that this norm is equivalent to the graph norm. Given a vector $u\in H(div;\Omega)$, it is possible to define its normal component u. n on the boundary [1].

LEMMA 1.1.

Let $\Omega\subset R_3$ be Lipschitz continuous. Then, the operator $\gamma_n: C^\infty(\overline{\Omega})^3\to C^\infty(\partial\Omega)^3$, mapping a vector into its normal component on the boundary, can be extended continuously to an operator $\gamma_n: H(div;\Omega)\to H^{-1/2}(\partial\Omega)^3$. In addition, there exists a continuous lifting operator $R_n: H^{-1/2}(\partial\Omega)\to H(div;\Omega)$, such that $\gamma_n(R_n\phi)=\phi,\phi\in H^{-1/2}(\partial\Omega)$. The following Green's formula holds, $u\in H(div;\Omega)$ and $q\in H^1(\Omega)$,

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$$\int_\Omega u.gradqdx + \int_\Omega divuqdx = \int_{\partial\Omega} u.nqdS.$$

where the integral on the right hand side is understood as the duality pairing between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$. We consider finite element discretization's based on the Raviart-Thomas elements. For triangulations made of tetrahedra, the local spaces on a generic tetrahedron K are defined as

$$D_k(K) := P_{k-1}(K)^n \oplus \times \tilde{P}_{k-1}(K),\ k \geq 1,$$

where x is the position vector in R^n and $\tilde{P}_{k-1}(K)$ is the space of homogeneous polynomials of

degree $k-1$ on K. A function u in $D_k(K)$ is uniquely defined by the following degrees of freedom

$$\int_f u.npdx,\ \mathrm{P} \in P_{k-1}(f),$$

for each face for each face f of K. For k > 1, we add

$$\int_k u.\mathrm{P}dx,\ \mathrm{P} \in P_{k-2}(K)^n$$

It can be proven that the following spaces are well-defined:

$$RT_k^h(\Omega) := \{u \in H(div;\Omega) \mid u|_{\mathrm{K}} \in D_k(K), \mathrm{K} \in \mathrm{T}_h\},$$

$$RT_{k;0}^h(\Omega) := \{u \in H_0(div;\Omega) \mid u|_{\mathrm{K}} \in D_k(K), \mathrm{K} \in \mathrm{T}_h\}$$

The corresponding interpolation operator is denoted by $\prod_{RT_k}^h$ When there is no ambiguity,

we simply use the notations $RT^h(\Omega),\ RT_0{}^h(\Omega),$ and $\prod_{RT}^h$

For the case k = 1, the elements of the local space have the simple form

$$D_1(K) = \{u = a + b\times \mid a \in P_0(K)^3, b \in P_0(K)\}$$

It is immediate to check that the normal components of a vector in $D_1(K)$ are constant on

each face f. These values

$$\lambda_f(u) = \frac{1}{|f|}\int_f u.nds,\quad f \subset \partial K,$$

can be taken as the degrees of freedom.

The spaces $H(curl;\Omega)$ and $H(div;\Omega)$, and special finite element approximations have been introduced to analyze equations such as (1.1); see [2]. In recent years, a considerable effort has been made to develop optimal or quasi-optimal, scalable preconditioners for these finite element approximations of problems in $H(curl;\Omega)$ and $H(div;\Omega)$. Two-level overlapping Schwarz preconditioners for these spaces have been developed for two (see [3]) and three (see [4-10]) dimensions, respectively. Multigrid and multilevel methods are considered in [11-28]. A few papers on the $H(curl;\Omega)$ case have proved optimality for a two-subdomain iterative sub structuring preconditioner, combined with Richardson's method, for a low-frequency approximation of time-harmonic Maxwell's equations in three dimensions. In [19] the authors restrict themselves to two dimensions and develop an iterative sub structuring method for the problems in space $H(curl;\Omega)$. In their work the condition number bound is independent of the number of substructures, it is developed locally for one substructure at a time and therefore insensitive to even large changes in the coefficients from one substructure to its neighbors.

Domain decomposition (DD) methods are important tools for solving partial differential equations, especially by parallel computers. In this paper, we shall study a class of nonoverlapping DD method, which is based on using Robin-Robin boundary conditions as transmission conditions on the subdomain interface. There is some iterative Robin-Robin DD method in the literature. This method was first proposed by Lions. Actually, the optimal transmission condition on the interface involves a nonlocal operator, which is not convenient to implement. So, adopting some local approximations to this operator becomes a useful way, which usually result in Robin-type boundary conditions. Some interesting Robin-type transmission conditions are considered in [21] for elliptic problem, and in [23] for the Helmholtz problem, respectively. Higher order approximation is also studied in [9], where they use the Pade approximates to localize the operator. However, the higher of approximation order, the more additional costs it has. The convergence was proved to be $1-O(h)$ in [16,25]. Then, in [29], Gander, Halpern, and Nataf improved it to be $1-O(h^{1/2})$ in the case of two subdomains. Later, Qin and Xu gave the convergence rate $1-O(h^{1/2}H^{1/2})$ in many subdomains. Later, Qin and Xu gave the convergence rate $1-O(h^{1/2}H^{1/2})$ in many subdomain cases in [30], and further they used the Dirichlet-to-Neumann operator to prove that the convergence rate cannot be improved. A sharp convergent result is also obtained in [31-35]. For the case of arbitrary geometry interface, Liu also got the convergence rate to be $1-O(h^{1/2})$ in [22]. For the second-order Robin-type transmission condition, the convergence rate was improved to $1-O(h^{1/4})$ in [36]. Recently, for the two subdomain case, a new two-parameter Robin-Robin DD method was proposed by Chen, Xu, and Zhang in [37]. It is shown that the convergence rate is independent of the mesh size h without any additional computation. For the discontinuous coefficients problem, the convergence of the optimized Schwarz method was analyzed in [19] for the two subdomains case. In [30,31], the authors introduced some preconditioned systems of the optimized Schwarz method. In [29], the authors present a new Robin-type nonoverlapping domain decomposition (DD) preconditioner, and they have proved the preconditioner is optimal and scalable.

In this paper, we restrict ourselves to three dimensions and develop a preconditioner for equation (1.1). The preconditioner is induced from the Robin-Robin domain decomposition (DD) algorithm. We have proved that the condition bound is independent of h, so the method is optimal. To analyze the bound of the condition number, it has been necessary to obtain a continuous extension from the trace space of $H(div;\Omega)$ into it, which has been constructed in [38]. The outline of this paper is as follows. In section 2, we will describe our Robin-Robin DD algorithm. In section 3, we will describe our preconditioner induced from the renew equation of the algorithm. In section 4, we will consider its corresponding condition number estimate. The implementation of this new DD algorithm will be discussed in section 5. Finally, some numerical results to verify our theoretical results will be given in the last section.

Model Problem and DD Algorithm

Problem (2.1) is equivalent to the following coupled problem, the equivalence can be proved

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by considering the corresponding variational problems.

$$\begin{cases}-grad(a_1 divu_1)+B_1u_1 = fin\Omega_1\\ u_1.n_1 = 0on\partial\Omega_1\cap\partial\Omega\\ u_1.n_1 = -u_2.n_2on\Gamma,\\ a_1divu_1 = a_2divu_2on\Gamma,\\ -grad(a_2divu_2)+B_2u_2 = fin\Omega_2\\ u_2.n_2on\partial\Omega_2\cap\partial\Omega\end{cases}$$

where the first transmission condition is chosen based on the fact that only the normal components of the finite element functions are continuous across the interelement boundaries, see [39,45]. The transmission conditions of (2.1) is equivalent to the following Robin-Robin boundary conditions:

$$\begin{cases}\gamma_1u_1.n_1+a_1divu_1 = \gamma_1u_2.n_2+a_2divu_2 = g_1on\Gamma,\\ \gamma_2u_2.n_2+a_2divu_2 = \gamma_2u_1.n_1+a_1divu_1 = g_2on\Gamma,\end{cases}$$

where we allow γ_1,γ_2 to any positive constants such that $\gamma_1 |= O(h),\gamma_2 = [o(1)]^{-1}$

Let $V_{i;0} = H_0(div;\Omega)\,|\,\Omega_{i,}$ multiply the first equation of (2.1) by $v_1\in V_{1;0,}$ and integrate it

over Ω_1, we get

$$\int_{\Omega_1}(-grad(a_1divu_1)+B_1u_1).v_1dx = \int_{\Omega_1} f.v_1dx,$$

by parts, with the Green's formula, we get

$$\int_{\Omega_1} a_1divu_1divv_1dx+\int_{\Omega_1} B_1u_1.v_1dx-\int a_1divu_1(v_1.n_1)ds = \int_{\Omega_1} f.v_1dx$$

Using the boundary condition (2.2), we get

$$a_{div}^{(1)}(u_1,v_1)+\gamma_1(\tilde{u}_1,\tilde{v}_1)_\Gamma = (f,v_1)_{\Omega_1}+\left\langle g_1,\tilde{v}_1\right\rangle_{\Gamma.}$$

This way, we get the second variational problem on Ω_2:

$$a_{div}^{(1)}(u_2,v_2)+\gamma_2(\tilde{u}_2,\tilde{v}_2)_\Gamma = (f,v_2)_{\Omega_2}+\left\langle g_2,\tilde{v}_2\right\rangle_{\Gamma.}$$

Where

$$a_{div}^{(i)}(u,v)\int_{\Omega_i}(a_idivudivv+B_iu.v)dx,\ \ i=1,2,$$

$$(f,v_i)_{\Omega_i} = \int_{\Omega_i} f.v_idx,\ i=1,2,$$

$$\left\langle\tilde{u},\tilde{v}\right\rangle_\Gamma = \int_\Gamma \tilde{u}.\tilde{v}ds,\ \tilde{u} = u.n.$$

Definition (The Robin-Robin DD method.) Given $g_2^0(=\theta)$ on Γ a serial version domain

decomposition iteration consists the following five steps (m=0,1, ...):

a) Solve on Ω_2 for u_{2h}^m

$$a_{div}^{(2)}(u_{2h}^m,v_{2h})+\gamma_2\left\langle\tilde{u}_{2h}^m,\tilde{v}_{2h}\right\rangle_\Gamma = (f,v2h)_{\Omega_2}-\left\langle g_2^m,v\tilde{2}h\right\rangle\Gamma\,|\,\forall_{v2h}\in|V_2$$

b) Update the interface condition on Γ:

$$\dot{g}_2^m = -g_1^m+(\gamma_1+\gamma_2)\tilde{u}_{1h.}^m$$

Get the next iterate by a relaxation:

$$g_2^{m+1} = \theta g_2^m+(1-\theta)\dot{g}_2^m.$$

The Preconditioner Induced from the DD Algorithm

Let $V_\Gamma^{(i)} := \{v_h.n_i\,|\,\partial\Omega_i\cap\Gamma\}$. Define $E_{ih}: V_\Gamma^{(i)}\to V_{ih}$ as follows:

$$\begin{cases}a_{div}^{(i)}(E_{ih}\tilde{w}_{ih},v_{ih}) = 0\forall v_{ih}\in V_{ih}^0\\ E_{ih}\tilde{w}_{ih}.n_i\,|\,\Gamma = \tilde{w}_{ih},\\ E_{ih}\tilde{w}_{ih}.n_i\,|\,\partial\Omega_i\cap\partial\Omega = 0,\end{cases}$$

Where $\tilde{w}_{ih} = w_h.n_i\,|\,\Gamma\in V_\Gamma^{(i)}$ and $E_i\tilde{w}_{ih}\in V_{ih}, i=1,2.$ Let u_{0ih} be the solution of the Dirichlet

problem:

$$\begin{cases}a_{div}^{(i)}(u_{0ih},v_{ih}) = f(v_{ih})\in V_{ih}^0,\\ u_{0ih}.n_i\,|\,\partial\Omega_i = 0.\end{cases}$$

Define S_{ih} to be a linear operator on V_Γ which satisfies the following equations:

$$V_\Gamma\left\langle S_{ih}\tilde{u}_{ih},\tilde{v}_{ih}\right\rangle_\Gamma := a_{div}^{(i)}(E_{ih}\tilde{u}_{ih},T_{ih}\tilde{v})\forall\tilde{v}_{ih}\in v_\Gamma^{(i)},$$

Where $T_{ih}: v_\Gamma^{(i)}\to v_{ih}$ is any extension such that $T_{ih}\tilde{v}_{ih}.n_i\,|\,\partial\Omega_i\cap\Gamma = \tilde{v}_{ih.}$ Define $\tilde{f}_{ih}$ satisfies

$$\left\langle\tilde{f}_{ih},\tilde{v}_{ih}\right\rangle_\Gamma := f(T_{ih}\tilde{v}_{ih})-a_{div}^{(i)}(u_{0ih},T_{ih}\tilde{v}_{ih})\ \forall V_{ih}\in v_\Gamma^{(i)}$$

The definition of s_{ih} and f_{ih} is independent of the choice of T_{ih} for all $v_{ih}\in v_T^{(i)}$.

Note that $E_i\mathbf{w}_eih+\mathbf{u}_{0ih}$ satisfying:

$$\begin{cases}a_{div}^{(i)}(E_i\tilde{w}_{ih}+u_{0ih},v_{ih}) = f(v_{ih})\\ (E_i\tilde{w}_{ih}+u_{0ih}).n_i\,|\,\partial\Omega_i\cap\tau = \tilde{w}_{ih}, \qquad v_{ih}\in v_0^{ih}\\ (E_i\tilde{w}_{ih}+u_{0ih}).n_i\,|\,\partial\Omega_i\cap\partial\Omega = 0\end{cases}$$

Based on Lax-Milgram theorem the solution of the problem is unique, i.e.

$$\mathbf{u}ih = Eih\mathbf{w}_eih+\mathbf{u}0ih,$$

so that,

$$hS_{ih}\mathbf{u}e_{ih},\mathbf{v}e_{ih}i\Gamma-he\mathbf{f}_{ih},\mathbf{v}e_{ih}i\Gamma = a_{div}^{(i)}(E_{ih}\mathbf{u}e_{ih},T_{ih}\mathbf{v}e_{ih})-(\mathbf{f}(T_{ih}\mathbf{v}e_{ih})-a^{(i)}{}_{div}(\mathbf{u}0_{ih},T_{ih}\mathbf{v}e_{ih}))$$

$$= a_{div}{}^{(i)}(E_{ih}\mathbf{u}e_{ih}+\mathbf{u}0_{ih},T_{ih}\mathbf{v}e_{ih})\ -\mathbf{f}(T_{ih}\mathbf{v}e_{ih})$$

$$= a_{div}{}^{(i)}(E_{ih}\mathbf{u}e_{ih}+\mathbf{u}0_{ih},\mathbf{v}_{ih})\ -\mathbf{f}(\mathbf{v}_{ih}) = a_{div}{}^{(i)}(\mathbf{u}ih,\mathbf{v}ih)-\mathbf{f}(\mathbf{v}ih).$$

Therefore (2.3) and (2.4) are equivalent to

$$\langle(S_{1h}+\gamma_1I)\mathbf{u}_{1h},\mathbf{v}_{1h}i\rangle r = \langle f_{1h},v_{1h}\rangle r+\langle g_1,v_{1h}\rangle r\forall v_{1h}\in V_{1h}$$

$$\langle(S_{2h}+\gamma_2I)u_{2h},v_{2h}\rangle r = \langle f_{2h},v_{2h}\rangle r-\langle g_2,v_{2h}\rangle r\forall v_{2h}.$$

Because of the of the arbitrariness of v_{ih}, we obtain

$$(S_{1h}+\gamma_1 I)u_{1h} = f_{1h}+g_1,$$
$$(S_{2h}+\gamma_2 I)u_2 h = \mathbf{f}_2 h - \mathbf{g}_2.$$

The Robin-Robin Iteration is equivalent to

$$\mathbf{u}^m{}_{2h} = (S_{2h}+\gamma_2 I)-1(\mathbf{f}_{2h}-\mathbf{g}_m^2).$$
$$\mathbf{u}^m{}_{1h} = (S_{1h}+\gamma_1 I)-1(\mathbf{f}_{1h}+\mathbf{g}_1{}^m).$$
$$\mathbf{g}_2^m = -\mathbf{g}_1{}^m + (\gamma_1+\gamma_2)\mathbf{u}^m{}_{1h}.$$
$$\mathbf{g}_2{}^{m+1} = \theta\mathbf{g}_2{}^m + (1-\theta)\mathbf{g}_2{}^m$$

We induce that

$$g_2^{m+1} = g_2^m + (1-\theta)P_r^{-1}(f_r - Gg_2^m),$$

Where

$$P_r^{-1} = (\gamma_2 I - S_{1h})(\gamma_1 I - S_{1h})^{-1h},$$
$$G = (\gamma_1 I - S_{1h})(\gamma_2 I - S_{1h})^{-1} - (\gamma_1 I - S_{2h})(\gamma_2 I - S_{2h})^{-1}$$
$$f_r = (\gamma_1-\gamma_2)(\gamma_1 I - S_{1h})^{-1}\tilde{f}_{1h} - (\gamma_1+\gamma_2)(\gamma_1 I - S_{2h})^{-1}\tilde{f}_{2h},$$

which shows that the renew equation of the Robin-Robin algorithm is a preconditioned Richarson iteration for the following equation:

$$Gg_2 = f_r,$$

where the preconditioned operator is P_r^{-1}

We notice that

$$[(\gamma_2 I - S_{1h})(\gamma_1 I - S_{1h})^{-1}]^T = (\gamma_2 I - S_{1h})^{-1}$$

In fact, $[(\gamma_2 I - S_{1h})(\gamma_1 I - S_{1h})^{-1}]^T = (\gamma_1 I - S_{1h})^{-1}(\gamma_2 I - S_{1h})$

$$= (\gamma_1 I - S_{1h})^{-1}((\gamma_1 I - \gamma_2)I - (\gamma_1 I -$$
$$= (\gamma_1-\gamma_2)(\gamma_1 I - S_{1h})^{-1} - 1$$
$$= (\gamma_1-\gamma_2)(\gamma_1 I - S_{1h})^{-1} - (\gamma_1 I - S_{1h})^{-1}(\gamma_1 I - S_{1h})$$
$$= -(\gamma_1 I - S_{1h})^{-1}(\gamma_2 I - S_{1h})$$

In the same way,

$$[(\gamma_1 I - S_{2h})(\gamma_2 I - S_{2h})^{-1}]^T = (\gamma_1 I - S_{2h})(\gamma_2 I - S_{2h})^{-1}$$

Thus, the operators G and P_r^{-1} are symmetric and P_r^{-1} is positive definite, G is also positive

definite will be proved later. Therefore we can use the preconditioned conjugate gradient (PCG)

method to solve the system.

The Condition Number Estimate

LEMMA 4.1. For the discrete harmonic extension operator E^h, we have (deduced from Theorem 2.5 in [32], or Lemma A.19 in [7])

$$\forall \tilde{w}_h \in V_\Gamma.$$

LEMMA 4.2: It holds that

$$\left\|\tilde{w}_h\right\|_{-\frac{1}{2};\partial\Omega} \le C\left\|E_h \tilde{w}\right\|_{div;\Omega},$$

$$\left\|E_h \tilde{w}\right\|_{0;T}^2 + H_T^2\left\|divE_h \tilde{w}\right\|_{0;T}^2 \le C\left\|\tilde{w}_h\right\|_{-\frac{1}{2};\partial T}^2$$

Proof. The proof is similar to one given in [24, Lemma 4.3]. We will first prove the result for a

substructure T of unit diameter. Consider a Neumann problem:

$$\begin{cases} -\Delta\Phi + = 0 \text{ in T}, \\ \dfrac{\partial\phi}{\partial n} = \tilde{w}_h\, on \partial T \end{cases}$$

Here $\frac{\partial}{\partial n}$ is the derivative in the direction of the outward normal of ∂T. Define an extension operator $H_h\, \tilde{w}_h := \rho_h u$, where u = $grade\phi$, and ρ_h is the interpolant onto the Raviart-Thomas space $X_h(T)$, defined by the degrees of freedom of X_h are given by the averages of the normal components over the faces of the triangulation:

$$\lambda_f\,(\mathrm{u}) := \frac{1}{|f|}\int_f u\cdot nd\mu, f\in F_h$$

Where $|f|$ is the area of the face f and the direction of the normal can be fixed arbitrarily for each face, we will show below that they $\{\lambda_f(\mathrm{u})\}$ are well defined. Using the surjectivity of the map $\phi \to \partial\phi$ onto $H^s(T)$ and a regularity result given in [9, Corollary 23.3], we deduce

$$\|\phi\|_{\frac{3}{2}+s;T} \le \|\tilde{w}_h\|_{s;\partial T,}\ S < \in_T$$

where $\in_T$ is strictly positive and depends on *T*.

$$\|u\|_{\frac{1}{2}+s;T} = \|\nabla\phi\|_{\frac{1}{2}+s;T} \le \|\phi\|_{\frac{3}{2}+s;T} \le \|\tilde{w}_h\|_{s;\partial T,}$$

$$\|divu\|_{\frac{1}{2}+s;T} = \|\Delta\phi\|_{\frac{1}{2}+s;T} = \|\phi\|_{\frac{1}{2}+s;T} \le \|\phi\|_{\frac{3}{2}+s;T} \le \left\|\tilde{w}_h\right\|_{s;\partial T.}$$

They $\{\lambda_f(\mathrm{u})\}, f\in F_h\}$ are well defined, since u $\in H^{\frac{1}{2}+s}(T)$, with $s > 0$, then

$$|E_h\tilde{w}_h\|_{div;T}^2 \le Ca_{div}(E_h\tilde{w}_h, E_h\tilde{w}_h)$$
$$\le Ca_{div}(H_h\tilde{w}_h, H_h\tilde{w}_h)$$
$$\le C\|H_h\tilde{w}_h\|_{div;T}^2$$
$$= C\|\rho_h u\|_{div;T}^2$$
$$= C\|\rho_h u - u) + \|u\|_{div;T}^2$$
$$\le C(\|(\rho_h u - u\|_{div;T}^2 + \|u\|_{div;T}^2),$$

Where $\tilde{a}_{div(u,v)} = \int_T (divudivv + u\cdot v)dx$, and R_n is the extension of Lemma in [3].

$$\|\rho_h u - u\|_{div;T}^2 = \|\rho_h u - u\|_{0;T}^2 + \|div(\rho_h u - u)\|_{0;T}^2$$

We find that

$$\|\rho_h u - u\|_{0;T} \le Ch^{\frac{1}{2}+s\|u\|\frac{1}{2}+s;T}$$
$$\le Ch^{\frac{1}{2}+s\|\tilde{w}_h\|s;\partial T}$$

$$\leq Ch^{\frac{1}{2}+s} h^{-(s-(-\frac{1}{2}))} \| \tilde{w}_h \|_{-\frac{1}{2};\partial T}$$

$\leq C \| \tilde{w}_h \|_{-\frac{1}{2};\partial T}$ and

$$\| div(\rho_h u - u \|^2_{0;T} = \sum_{t \in T} \| div(\rho_h u - u) \|^2_{0;t}$$

$$\sum_{t \in T} \| div(\rho_h u - divu) \|^2_{0;t}$$

$$\sum_{t \in T} \| \Pi_h divu - divu) \|^2_{0;t},$$

$$\||| pu \|_{0;t} = \| u - \frac{1}{|t|} \int_t udx \|_{0;t}$$

$$\||| P \|_{H^{\frac{1}{2}+s}(t) \to H^0(t)} \leq \| P \|^{1-\theta}_{H^0(t) \to H^0(t)} \| P \|^{\theta}_{H^1(t) \to H^0(t)}$$

$$\||| \Pi_h diuv - divu \|_{0,t} \leq ch_t^{\frac{1}{2}+s} \| diuv \|_{\frac{1}{2}+s,t.}$$

Where, $\leq ch_t \| u \|_{1;t}$,

is the L^2 projection onto the space of constant functions on each fine $t \in T$, and Π_h, $\Pi_h divu = \frac{1}{|t|} \int_t divudx$, $\forall t \in T$.

Let $P: u \to u - \frac{1}{|t|} \int_t udx$, then

$$\| pu \|_{0;t} = \| u - \frac{1}{|t|} \int_t udx \|_{0;t}$$

$\leq 2 \| u \|_{0;t}$ and

$$\| pu \|_{0;t} = \| u - \frac{1}{|t|} \int_t udx \|_{0;t}$$

$$\leq ch_t \mid u \mid_{1;t}$$

$$\leq ch_t \| u \|_{1;t},$$

thus,

$$\| P \| H^0(t) \to H^0(t) \leq 2,$$

$$\| P \| H^1(t) \to H^0(t) \leq Ch_t.$$

By interpolation theory, we have

$$\| P \|_{H^{\frac{1}{2}+s}(t) \to H^0(t)} \leq \| P \|^{1-\theta}_{H^0(t) \to H^0(t)} \| P \|_{0,t} \leq ch_t^{\frac{1}{2}+s} \| u \|_{\frac{1}{2}+s,t}.$$

$$\frac{1}{2} + s = (1-\theta) \cdot 0 + \theta \cdot 1.$$

i.e.

$\| P \|_{0,t} \leq ch_t^{\frac{1}{2}+s} \| u \|_{\frac{1}{2}+s,t}$, where

$\frac{1}{2} + s = (1-\theta) \cdot 0 + \theta \cdot 1$. Thus,

$$\| \Pi_h diuv - divu \|_{0,t} \leq ch_t^{\frac{1}{2}+s} \| diuv \|_{\frac{1}{2}+s,t.}$$

Employing an inverse inequality, we find that for $s < \in T$,

$$\|div(\rho_h u .. u)\|^2_0;T = \sum_{t \in T} ch_t^{2(\frac{1}{2}+s)} \| divu \|^2_{\frac{1}{2}+s;t}$$

$$\leq ch^{2(\frac{1}{2}+s)} \sum_{t \in T} \| divu \|^2_{\frac{1}{2}+s;t}$$

$$= ch^{2(\frac{1}{2}+s)} \sum_{t \in T} \| divu \|^2_{\frac{1}{2}+s;T}$$

$$\leq ch^{2(\frac{1}{2}+s)} ch^{-2(s-(\frac{1}{2}))} \| \tilde{w} h \|^2_{-\frac{1}{2};\partial T}$$

$$\leq C \| \tilde{w}_h \|^2_{-\frac{1}{2};\partial T}$$

therefore, $\|E_h \tilde{w}_h \|div;T \leq C \| \tilde{w}_h \|_{-\frac{1}{2};\theta T}$ We now consider a substructure T of diameter H, obtained by dilation from a sub structure of unit diameter. Using the previous result and by using a scaling argument, we have

$$\|E_h \tilde{w}_h \|^2_0 \;;T + H_T^2 \|divE_h \tilde{w}_h \|^2_0;T \leq C \| \tilde{w}_h \|^2_{-\frac{1}{2};\theta T}$$

Theorem 4.3.

$$a^{(i)}_{div}(E_{ih} \tilde{w}_h ; E_{ih} \tilde{w} h) \leq C \| \tilde{w}_h \|^2_{0;r}, i = 1,2..$$

Proof: $\forall \tilde{w}_h \in v_r$, we have

$$a^{(i)}_{div}(E_{ih} \tilde{w}_h ; E_{ih} \tilde{w}_h) = a^{(i)}_{div}(E_{ih} \tilde{w}_h ; E_{ih} \tilde{w}_h)$$

$$C \| E_{ih} \tilde{w}_h \|^2_{div} ; \Omega_i$$

$$\leq C \| \tilde{w}_h \|^2_{-\frac{1}{2};\theta\Omega_i}$$

$$= C(\sup \frac{\langle w_{ih}, \phi \rangle}{\| \phi \|_{-\frac{1}{2};\theta\Omega_i}}, \phi \in H_0^{\frac{1}{2}}(\theta\Omega_i), \phi \neq 0$$

we find that, $\forall \phi : \phi \in H_0^{\frac{1}{2}}(\partial\Omega_i), \phi \neq \theta$, we have,

$$\langle \tilde{w}_h, \phi \rangle = \int_{\theta\Omega_i} \tilde{w}_h \phi d\sigma$$

$$\leq (\int_{\theta\Omega_i} \tilde{w}_h \phi d\sigma)^{\frac{1}{2}} (\int_{\theta\Omega_i} \phi^2 d\sigma)^{\frac{1}{2}}$$

$$=\| \tilde{w}_{ih} \| 0;\theta\Omega_i \ \| \theta \| \ 0;\theta\Omega_i$$

And

$$\frac{\langle \tilde{w}_{ih,}\phi\rangle}{\|\phi\|_{-\frac{1}{2};\theta\Omega_i}} \le \frac{\| w_{ih,} \| 0;\theta\Omega_i \ \|\phi\| 0;\theta\Omega_i}{\|\phi\|_{-\frac{1}{2};\theta\Omega_i}}$$

$$\le \| w_{ih,} \| 0;\theta\Omega_i$$

$$\le \| w_{ih,} \| 0;\tau$$

$$=\| \tilde{w}_h \| 0;\tau$$

Therefore

$$a_{div}^{(i)}(E_{ih}\tilde{w}_h;E_{ih}\tilde{w}h) \le C\|\tilde{w}_h\|_{0;r}^2, i=1,2..$$

Theomem 4.4. It holds that

$$\|\tilde{w}_h\|_{0;\tau}^2 \le Ch^{-1}(a_{div}^{(i)}(E_{ih}\tilde{w}_h;E_{ih}\tilde{w}_h)+a_{div}^{(2)}(E_{2h}\tilde{w}_h;E_{2h}\tilde{w}_h))$$

Proof we find that,

$$\frac{\|\tilde{w}_h\|_{0;\tau}^2}{\|\tilde{w}_h\|_{\frac{1}{2};\tau}^2} = \frac{\|\tilde{w}_h\|_{0;\partial\Omega_i}^2}{\|\tilde{w}_h\|_{\frac{1}{2};\partial\Omega_i}^2}$$

$$=\frac{\langle \tilde{w}_h,\tilde{w}_h\rangle}{\|\tilde{w}_h\|_{-\frac{1}{2};\theta\Omega_i}^2}$$

$$\le \sup\frac{\langle \tilde{w}_h,\phi\rangle}{\|\phi\|_{-\frac{1}{2};\theta\Omega_i}}$$

$$=\|\tilde{w}_h\|_{-\frac{1}{2};\partial\Omega_i}$$

By inverse inequality, we have

$$\|\tilde{w}_h\|_{\frac{1}{2};\tau} \le Ch^{-(\frac{1}{2}-0)}\|\tilde{w}_h\| 0;\tau$$

$$= Ch^{-\frac{1}{2}}\|\tilde{w}_h\| 0;\tau \tag{4.3}$$

Combining (4.2), (4.3), and Lemma 4.2, we obtain

$$\|\tilde{w}_h\|_{0;\tau}^2 \le Ch^{-1}\|\tilde{w}_h\|_{-\frac{1}{2};\theta\Omega_i}^2$$

$$\le Ch^{-1}\| E_{ih}\tilde{w}_h\|_{div;\Omega_i}^2, i=1,2 \tag{4.4}$$

The final result is obtained by summing (4.4) over i = 1, 2

Theomem 4.5:

$$a_{div}^{(2)}(E_{2h}\tilde{w}_h,E_{2h}\tilde{w}_h) \le Ca_{div}^{(1)}(E_{1h}\tilde{w}_h,E_{1h}\tilde{w}_h)\cdot$$

Proof. Based on Lemmas 4.1, 4.2, and the equivalence of the energy norm and graph norm, we

have,

$$a_{div}^{(2)}(E_{2h}\tilde{w}_h,E_{2h}\tilde{w}_h) = a_{div}^{(2)}(E_{2h}\tilde{w}_{2h},E_{2h}\tilde{w}_{2h})$$

$$\le C\| E_{2h}\tilde{w}_h\|_{div;\Omega_2}^2$$

$$\le C\|\tilde{w}_{2h}\|_{-\frac{1}{2;\theta\Omega_2}}^2$$

$$= C\|\tilde{w}_{2h}\|_{-\frac{1}{2};\tau}^2$$

$$= C\|\tilde{w}_{1h}\|_{-\frac{1}{2};\tau}^2$$

$$\le C\| E_{1h}\tilde{w}_h\|_{div;\Omega_1}^2$$

$$\le Ca_{div}^{(1)}(E_{1h}\tilde{w}_{1h},E_{1h}\tilde{w}_{1h})$$

$$\le Ca_{div}^{(1)}(E_{1h}\tilde{w}_{1h},E_{1h}\tilde{w}_{1h})$$

LEMMA 4.6. It holds that

$$\frac{\gamma 1}{\gamma 2}\|\tilde{w}_h\|_{0,\Gamma}^2 + \frac{1}{\gamma 2}a_{div}^{(1)}(E_{1h}\tilde{w}_h,E_{1h}\tilde{w}_h) \le \left\langle(\gamma_1 I+S_{1h})(\gamma_2 I-S_{1h})(\gamma_2 I-S_{1h})^{-1}\tilde{w}_h,\tilde{w}_h,\right\rangle\Gamma$$

$$\le \frac{\gamma 1}{\gamma 2}\|\tilde{w}_h\|_{0,\Gamma}^2 + \frac{2}{\gamma 2}a_{div}^{(1)}(E_{1h}\tilde{w}_h,E_{1h}\tilde{w}_h)\forall \tilde{w}_h \in V_\Gamma .$$

Proof. Define $\tilde{e}_{1h} := (\gamma_2 I - S_{1h})^{-1}\tilde{w}_{1h}$, it is easy to check that

$$\langle(\gamma_1 I+S_{1h})(\gamma_2 I+S_{1h})^{-1}\tilde{w}_{1h},\tilde{w}_{1h}\rangle\Gamma$$
$$=\langle(\gamma_1 I+S_{1h})(\gamma_2 I+S_{1h})^{-1}\tilde{w}_{1h},\tilde{w}_{1h}\rangle\Gamma$$
$$=\langle(\gamma_1 I+S_{1h})\tilde{e}_{1h},(\gamma_2 I+S_{1h})\tilde{e}_{1h}\rangle\Gamma$$
$$=\gamma_1\gamma_2\|\tilde{e}_{1h}\|_{0;\Gamma}^2 + (\gamma_2-\gamma_1)\left\langle S_{1h}\tilde{e}_{1h},\tilde{e}_{1h}\right\rangle\Gamma - \left\langle S_{1h}\tilde{e}_{1h},S_{1h}\tilde{e}_{1h}\right\rangle\Gamma$$

$$\|\tilde{w}_h\|_{0,\Gamma}^2 = \|(\gamma_2 I-S_{1h})\tilde{e}_{1h}\|_{0,\Gamma}^2 = \gamma_2^2\left\langle\tilde{e}_{1h},\tilde{e}_{1h}\right\rangle_\Gamma - 2\gamma_2\left\langle S_{1h}\tilde{e}_{1h},\tilde{e}_{1h}\right\rangle_\Gamma + \left\langle S_{1h}\tilde{e}_{1h},S_{1h}\tilde{e}_{1h}\right\rangle_\Gamma \tag{4.7}$$

$$a_{div}^{(1)}(E_{1h}\tilde{w}_h,E_{1h}\tilde{w}_h)$$
$$=a_{div}^{(1)}(E_{1h}\tilde{w}_h,E_{1h}\tilde{w}_h)$$
$$=a_{div}^{(1)}(E_{1h}((\gamma_2 I-S_{1h})\tilde{e}_{1h}),E_{1h}((\gamma_2 I-S_{1h})\tilde{e}_{1h}))$$
$$=\gamma_2^2 a_{div}^{(1)}(E_{1h}\tilde{e}_{1h},E_{1h}\tilde{e}_{1h}) - 2\gamma_2 a_{div}^{(1)}(E_{1h}((S_{1h}\tilde{e}_{1h}),E_{1h}\tilde{e}_{1h}) + a_{div}^{(1)}(E_{1h}(S_{1h}\tilde{e}_{1h}),E_{1h}(S_{1h}\tilde{e}_{1h})). \tag{4.8}$$

Based on the definitions in section 3, we have,

$$\langle \tilde{e}_{1h}, \tilde{e}_{1h} \rangle_\Gamma = \| \tilde{e}_{1h} \|^2_{0,\Gamma},$$

$$\langle S_{1h}\tilde{e}_{1h}, S_{1h}\tilde{e}_{1h} \rangle_\Gamma = \| S_{1h}\tilde{e}_{1h} \|^2_{0,\Gamma},$$

$$\langle S_{1h}\tilde{e}_{1h}, \tilde{e}_{1h} \rangle_\Gamma = a^{(1)}_{div}(E_{1h}\tilde{e}_{1h}, T_{1h}\tilde{e}_{1h})$$

$$= a^{(1)}_{div}(E_{1h}\tilde{e}_{1h}, E_{1h}\tilde{e}_{1h}),$$

$$a^{(1)}_{div}(E_{1h}(S_{1h}\tilde{e}_{1h}), E_{1h}\tilde{e}_{1h}) = a^{(1)}_{div}(E_{1h}\tilde{e}_{1h}, E_{1h}(S_{1h}\tilde{e}_{1h}))$$

$$= \langle S_{1h}\tilde{e}_{1h}, S_{1h}\tilde{e}_{1h} \rangle_\Gamma$$

$$= \| S_{1h}\tilde{e}_{1h} \|^2_{0,\Gamma}.$$

Combine (4.6), (4.7), (4.8) with (4.9), (4.10), (4.11), (4.12), we obtain,

$$\langle (\gamma_2 I + S_{1h})(\gamma_2 I - S_{1h})^{-1}\tilde{w}_h, \tilde{w}_h \rangle_\Gamma$$

$$= \gamma_1\gamma_2 \| \tilde{e}_{1h} \|^2_{0,\Gamma} + (\gamma_2 - \gamma_1) a^{(1)}_{div}(E_{1h}\tilde{e}_{1h}, E_{1h}\tilde{e}_{1h}) - \| S_{1h}\tilde{e}_{1h} \|^2_{0,\Gamma}$$

$$= \gamma_2^2 \| \tilde{e}_{1h} \|^2_{0,\Gamma} - 2\gamma_2 a^{(1)}_{div}(E_{1h}\tilde{e}_{1h}, E_{1h}\tilde{e}_{1h}) + \| S_{1h}\tilde{e}_{1h} \|^2_{0,\Gamma}$$

$$= \gamma_2^2 a^{(1)}_{div}(E_{1h}\tilde{e}_{1h}, E_{1h}\tilde{e}_{1h}) - 2\gamma_2 \| S_{1h}\tilde{e}_{1h} \|^2_{0,\Gamma} + a^{(1)}_{div}(E_{1h}(S_{1h}\tilde{e}_{1h}), E_{1h}(S_{1h}\tilde{e}_{1h})).$$

Thus we have,

$$\langle (\gamma_1 I + s_{1h})(\gamma_2 I + s_{1h})^{-1}\tilde{w}_h, \tilde{w}_h \rangle \tau - \left(\frac{\gamma_1}{\gamma_2} \| \tilde{w}_h \|^2_{0;\tau} + \frac{1}{\gamma_2} a^{(1)}_{div}(E_{1h}\tilde{w}_h, E_{1h}\tilde{w}_h)\right)$$

$$\gamma_1 a^{(1)}_{div}(E_{1h}\tilde{e}_{1h}, E_{1h}\tilde{e}_{1h}) + \left(1 - \frac{\gamma_1}{\gamma_2}\right) \| s_{1h}\tilde{e}_{1h} \|^2_{0;\tau} - \frac{1}{\gamma_2} a^{(1)}_{div}(E_{1h}(s_{1h}\tilde{e}_{1h}), E_{1h}(s_{1h}\tilde{e}_{1h}))$$

$$\geq \frac{\gamma_2 - \gamma_1 - C}{\gamma_2} \| s_{1h}\tilde{e}_{1h} \|^2_{0;\tau} + \gamma_1 a^{(1)}_{div}(E_{1h}\tilde{e}_{1h}, E_{1h}\tilde{e}_{1h})$$

$$\geq 0$$

that is the left part of (4.5). On the other hand,

$$\| s_{1h}\tilde{e}_{1h} \|^2_{0;\tau} = a^{(1)}_{div}(E_{1h}(s_{1h}\tilde{e}_{1h}), E_{1h}(s_{1h}\tilde{e}_{1h}))$$

$$\leq a^{(1)}_{div}(E_{1h}(s_{1h}\tilde{e}_{1h}), E_{1h}(s_{1h}\tilde{e}_{1h}))^{\frac{1}{2}} a^{(1)}_{div}(E_{1h}(s_{1h}\tilde{e}_{1h}), E_{1h}(s_{1h}\tilde{e}_{1h}))^{\frac{1}{2}}$$

$$\leq a^{(1)}_{div}(E_{1h}(s_{1h}\tilde{e}_{1h}), E_{1h}(s_{1h}\tilde{e}_{1h}))^{\frac{1}{2}} C \| s_{1h}\tilde{e}_{1h} \|^2_{0;\tau}$$

Thus,

$$\| s_{1h}\tilde{e}_{1h} \|^2_{0;\tau} \leq C a^{(1)}_{div}(E_{1h}(s_{1h}\tilde{e}_{1h}), E_{1h}(s_{1h}\tilde{e}_{1h}))$$

Finally, based on the above inequality, we have

$$\frac{\gamma_1}{\gamma_2} \| \tilde{w}_h \|^2_{0;\tau} + \frac{2}{\gamma_2} a^{(1)}_{div}(E_{1h}\tilde{w}_h, E_{1h}\tilde{w}_h) - \langle (\gamma_1 I + s_{1h})$$

$$(\gamma_2 I + s_{1h})^{-1}\tilde{w}_h, \tilde{w}_h \rangle \tau$$

$$= (\gamma_2 - \gamma_1) a^{(1)}_{div}(E_{1h}\tilde{e}_{1h}, E_{1h}\tilde{e}_{1h}) - \left(3 - \frac{\gamma_1}{\gamma_2} C a^{(1)}_{div}(E_{1h}\tilde{e}_{1h}, E_{1h}\tilde{e}_{1h}) + \right.$$

$$\frac{2}{\gamma_2} a^{(1)}_{div}(E_{1h}(s_{1h}\tilde{e}_{1h}), E_{1h}(s_{1h}\tilde{e}_{1h}))$$

$$\geq \left((\gamma_2 - \gamma_1) - \left(3 - \frac{\gamma_1}{\gamma_2}\right) C\right) a^{(1)}_{div}(E_{1h}\tilde{e}_{1h}, E_{1h}\tilde{e}_{1h}) + \frac{2}{\gamma_2} a^{(1)}_{div}(E_{1h}(s_{1h}\tilde{e}_{1h}), E_{1h}(s_{1h}\tilde{e}_{1h}))$$

$$\geq 0$$

In the above inequality, we use the fact that $(\gamma_2 - \gamma_1) - (3 - \frac{\gamma_1}{\gamma_2})C > 0$ for sufficient small h. So

we have proved the right part of (4.5).

LEMMA 4.7. It holds that

$$\frac{\gamma_1}{\gamma_2} \| \tilde{w}_h \|^2_{0;\tau} - \frac{\gamma_1 + \gamma_2}{\gamma_1^2} a^{(1)}_{div}(E_{2h}\tilde{w}_h, E_{2h}\tilde{w}_h) \leq \langle (\gamma_1 I + s_{1h})(\gamma_2 I + s_{1h})^{-1}$$

$$\tilde{w}_h, \tilde{w}_h \rangle \tau \leq \frac{\gamma_1}{\gamma_2} \| \tilde{w}_h \|^2_{0;\tau} - \frac{\gamma_1 + \gamma_2}{(2+\delta)\gamma_2^2} a^{(1)}_{div}(E_{2h}\tilde{w}_h, E_{2h}\tilde{w}_h) \forall \tilde{w}_h \in V_r$$

Where,

$$\delta = \frac{C}{\gamma_2} = \frac{C}{[0(1)]^{-1}} << 0 \tag{4.13}$$

Proof. Define $\tilde{e}_{2h} := (\gamma_2 I + s_{1h})^{-1}\tilde{w}_h$, it is easy to check that

$$\langle (\gamma_1 I + s_{1h})(\gamma_2 I + s_{1h})^{-1}\tilde{w}_h, \tilde{w}_h \rangle \tau$$

$$= \langle (\gamma_1 I + s_{1h})\tilde{e}_{2h}, (\gamma_2 I + s_{1h})\tilde{e}_{2h} \rangle \tau \tag{4.14}$$

$$= \gamma_1\gamma_2 \langle \tilde{e}_{2h}, \tilde{e}_{2h} \rangle \tau + (\gamma_1 - \gamma_2)\langle s_{2h}\tilde{e}_{2h}, \tilde{e}_{2h} \rangle \tau - \langle s_{2h}\tilde{e}_{2h}, \tilde{e}_{2h} \rangle \tau,$$

$$\| \tilde{w}_h \|^2_{0;\tau} = \langle \tilde{w}_h, \tilde{w}_h \rangle \tau$$

$$= \langle (\gamma_2 I + s_{2h})\tilde{e}_{2h}, (\gamma_2 I + s_{2h})\tilde{e}_{2h} \rangle \tau \tag{4.15}$$

$$= \gamma_2^2 \langle \tilde{e}_{2h}, \tilde{e}_{2h} \rangle \tau + 2\gamma_2 \langle s_{2h}\tilde{e}_{2h}, \tilde{e}_{2h} \rangle \tau + \langle s_{2h}\tilde{e}_{2h}, \tilde{e}_{2h} \rangle \tau$$

$$a^2_{div}(E_{2h}\tilde{w}_h, E_{2h}\tilde{w}_h)$$

$$= a^2_{div}(E_{2h}((\gamma_2 I + s_{2h})\tilde{e}_{2h}), E_{2h}((\gamma_2 I + s_{2h})\tilde{e}_{2h}))$$

$$= \gamma_2^2 a^2_{div}(E_{2h}\tilde{e}_{2h}, E_{2h}\tilde{e}_{2h}) + 2\gamma_2 a^2_{div}(E_{2h}(s_{2h}\tilde{e}_{2h}), E_{2h}\tilde{e}_{2h}) + a^2_{div}(E_{2h}(s_{2h}\tilde{e}_{2h}),$$

$$E_{2h}(s_{2h}\tilde{e}_{2h}))$$

Based on the definitions in section 3, we have,

$$\langle \tilde{e}_{2h}, \tilde{e}_{2h} \rangle \tau = \| \tilde{e}_{2h} \|^2_{0;\tau} \tag{4.17}$$

$$\langle s_{2h}\tilde{e}_{2h}, \tilde{e}_{2h} \rangle \tau = a^2_{div}(E_{2h}\tilde{e}_{2h}, T_{2h}\tilde{e}_{2h})$$

$$= a^2_{div}(E_{2h}\tilde{e}_{2h}, E_{2h}\tilde{e}_{2h}) \tag{4.18}$$

$$\langle s_{2h}\tilde{e}_{2h}, s_{2h}\tilde{e}_{2h} \rangle \tau = \| s_{2h}\tilde{e}_{2h} \|^2_{0;\tau} \tag{4.19}$$

$$a^2_{div}(E_{2h}(s_{2h}\tilde{e}_{2h}), E_{2h}\tilde{e}_{2h}) = a^2_{div}(E_{2h}\tilde{e}_{2h}, E_{2h}(s_{2h}\tilde{e}_{2h})) \tag{4.20}$$

$$= \langle s_{2h}\tilde{e}_{2h}, s_{2h}\tilde{e}_{2h} \rangle \tau$$

$$=\| s_{2h} \tilde{e}_{2h} \|_{0;\tau}^2$$

Combine (4.14), (4.15), (4.16) with (4.17), (4.18), (4.19), (4,20), we obtain,

$$\langle(\gamma_1 I - s_{2h})(\gamma_2 I + s_{2h})^{-1} \tilde{w}_h, \tilde{w}_h\rangle\tau \quad (4.21)$$

$$= \gamma_1\gamma_2 \| \tilde{e}_{2h} \|_{0;\tau}^2 + (\gamma_1 - \gamma_2) a_{div}^2 (E_{2h} \tilde{e}_{2h}, E_{2h} \tilde{e}_{2h}) - \| s_{2h} \tilde{e}_{2h} \|_{0;\tau}^2$$

$$\| \tilde{w}_h \|_{0;\tau}^2 = \gamma_2^2 \| \tilde{e}_{2h} \|_{0;\tau}^2 + 2\gamma_2 a_{div}^2 (E_{2h} \tilde{e}_{2h}, E_{2h} \tilde{e}_{2h}) + \| s_{2h} \tilde{e}_{2h} \|_{0;\tau}^2 \quad (4.22)$$

$$a_{div}^2 (E_{2h} \tilde{w}_h, E_{2h} \tilde{w}_h) = \gamma_2^2 a_{div}^2 (E_{2h} \tilde{e}_{2h}, E_{2h} \tilde{e}_{2h}) + 2\gamma_2 \| s_{2h} \tilde{e}_{2h} \|_{0;\tau}^2 + a_{div}^2 (E_{2h}(s_{2h} \tilde{e}_{2h}), E_{2h}(s_{2h} \tilde{e}_{2h})) \quad (4.23)$$

Applying Lemma 4.3 and (4.21), (4.22), (4.23), we have,

$$\langle(\gamma_1 I - s_{2h})(\gamma_2 I + s_{2h})^{-1} \tilde{w}_h, \tilde{w}_h\rangle\tau - \left(\frac{\gamma_1}{\gamma_2} \| \tilde{w}_h \|_{0;\tau}^2 - \frac{\gamma_1+\gamma_2}{\gamma_2^2} a_{div}^2 (E_{2h} \tilde{w}_h, E_{2h} \tilde{w}_h)\right)$$

$$= \left((\gamma_1 - \gamma_2) - \frac{\gamma_1}{\gamma_2}.2\gamma_2 + \frac{\gamma_1+\gamma_2}{\gamma_2^2}.\gamma_2^2\right) a_{div}^2 (E_{2h} \tilde{e}_{2h}, E_{2h} \tilde{e}_{2h}) + \left(-1 - \frac{\gamma_1}{\gamma_2}.1 + \frac{\gamma_1+\gamma_2}{\gamma_2^2}.2\gamma_2^2\right)$$

$$\| s_{2h} \tilde{e}_{2h} \|_{0;\tau}^2 + \frac{\gamma_1+\gamma_2}{\gamma_2^2} a_{div}^2 (E_{2h}(s_{2h} \tilde{e}_{2h}), E_{2h}(s_{2h} \tilde{e}_{2h}))$$

$$= \left(1 + \frac{\gamma_1}{\gamma_2}\right) \| s_{2h} \tilde{e}_{2h} \|_{0;\tau}^2 + \frac{\gamma_1+\gamma_2}{\gamma_2^2} a_{div}^2 (E_{2h}(s_{2h} \tilde{e}_{2h}), E_{2h}(s_{2h} \tilde{e}_{2h}))$$

$$\geq 0$$

that is the left part of (4.13). On the other hand, we have

$$\langle(\gamma_1 I - s_{2h})(\gamma_2 I + s_{2h})^{-1} \tilde{w}_h, \tilde{w}_h\rangle\tau - \left(\frac{\gamma_1}{\gamma_2} \| \tilde{w}_h \|_{0;\tau}^2 - \frac{\gamma_1+\gamma_2}{(2+\delta)\gamma_2^2} a_{div}^2 (E_{2h} \tilde{w}_h, E_{2h} \tilde{w}_h)\right)$$

$$= \left((\gamma_1 - \gamma_2) - \frac{\gamma_1}{\gamma_2}.2\gamma_2 + \frac{\gamma_1+\gamma_2}{(2+\delta)\gamma_2^2}.\gamma_2^2\right) a_{div}^2 (E_{2h} \tilde{w}_h, E_{2h} \tilde{w}_h) + \left(-1 - \frac{\gamma_1}{\gamma_2}.1 + \frac{\gamma_1+\gamma_2}{\gamma_2^2}.2\gamma_2^2\right) \| s_{2h} \tilde{e}_{2h} \|_{0;\tau}^2$$

$$= -\frac{1+\delta}{2+\delta}(\gamma_1+\gamma_2) a_{div}^2 (E_{2h} \tilde{w}_h, E_{2h} \tilde{w}_h) - \frac{\delta}{2+\delta} \frac{\gamma_1+\gamma_2}{\gamma_2^2} \| s_{2h} \tilde{e}_{2h} \|_{0;\tau}^2 + \frac{\gamma_1+\gamma_2}{(2+\delta)\gamma_2^2} a_{div}^2 (E_{2h}(s_{2h} \tilde{e}_{2h}), E_{2h}(s_{2h} \tilde{e}_{2h}))$$

$$\leq 0$$

so we have proved the right part of (4.13). Based on Lemmas 4.6 and 4.7, we have Lemma 4.8.

LEMMA 4.8. It holds that

$$\frac{\gamma_1}{\gamma_2} \| \tilde{w}_h \|_{0;\tau}^2 + \frac{1}{\gamma_2} a_{div}^2 (E_{1h} \tilde{w}_h, E_{1h} \tilde{w}_h) \leq \langle \mathrm{P_r} \tilde{w}_h, \tilde{w}_h\rangle\tau$$

$$\leq \frac{\gamma_1}{\gamma_2} \| \tilde{w}_h \|_{0;\tau}^2 + \frac{2}{\gamma_2} a_{div}^2 (E_{1h} \tilde{w}_h, E_{1h} \tilde{w}_h) \forall \tilde{w}_h \in V_r$$

and

$$\frac{1}{\gamma_2} a_{div}^2 (E_{1h} \tilde{w}_h, E_{1h} \tilde{w}_h) + \frac{\gamma_1+\gamma_2}{(2+\delta)\gamma_2^2} a_{div}^2 (E_{2h} \tilde{w}_h, E_{2h} \tilde{w}_h) \leq (G \tilde{w}_h, \tilde{w}_h)\tau$$

$$\leq \frac{2}{\gamma_2} a_{div}^2 (E_{1h} \tilde{w}_h, E_{1h} \tilde{w}_h) + \frac{\gamma_1+\gamma_2}{\gamma_2^2} a_{div}^2 (E_{2h} \tilde{w}_h, E_{2h} \tilde{w}_h) \forall \tilde{w}_h \in V_r$$

Based on Lemmas 4.2, 4.3 and 4.8, we get the following main result of this paper.

THEOMEM 4.9. Assume that $\gamma_1 = O(h), \gamma_2 = [o(1)^{-1}]$ then,

$$C\langle \mathrm{P_r} \tilde{w}_h, \tilde{w}_h\rangle\tau \leq (G \tilde{w}_h, \tilde{w}_h)\tau \leq C\langle \mathrm{P_r} \tilde{w}_h, \tilde{w}_h\rangle\tau$$

Numerical Results

In this section, the convergence behavior of our new DD method will be checked by some numerical experiments. Consider the problem

$$\begin{cases} grad(divu) + u = f, in\Omega \\ u.n = \theta, on\theta\Omega \end{cases} \quad (5.1)$$

Where, $\Omega = [-1,1]\times[-1,1]\times[-1,1]$ and the function

$$f = [(1-\pi^2)\sin(\pi x), (1-\pi^2)\sin(\pi y), (1-\pi^2)\sin(\pi z)]^T$$

The exact solution of this problem is $u = [(\sin(\pi x), \sin(\pi y), \sin(\pi z)]^T$ The domain Ω is divided into 2 subdomains. We set the Robin parameters to be $\gamma_1 = h, 2h$, $\gamma_2 = 50, 100$ which satisfy the condition of Theorem 4.9. The iteration stops when $\frac{\| u_{iter} - u_d \|_{l^2}}{\| u_d \|_{l^2}} \leq 10^{-10}$ (Figures 1-4) and (Tables 1-4).

Table 1: Iterative number of γ1 = h and γ2 = 100.

h	1	1/2	1/4	1/8	1/16	1/32
CG	3	18	66	157	341	684
PCG	2	4	5	5	6	6

Table 2: Iterative number of γ1 = 2*h* and γ2 = 100.

h	1	1/2	1/4	1/8	1/16	1/32
CG	3	18	65	157	342	686
PCG	2	4	5	6	6	6

Table 3: Iterative number of γ1 = h and γ2 = 50.

h	1	1/2	1/4	1/8	1/16	1/32
CG	3	18	66	157	341	684
PCG	2	4	5	5	6	6

Table 4: Iterative number of γ1 = 2h and γ2 = 50.

h	1	1/2	1/4	1/8	1/16	1/32
CG	3	18	65	157	342	684
PCG	2	4	5	5	5	6

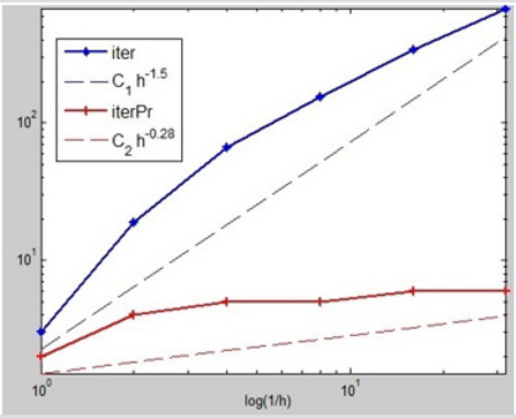

Figure 1: The figure describes the case of γ1 = h and γ2 = 100, where the ordinate values of these nodes on these polylines are iterative numbers which are the dates from Table 1, while the abscissa values of these nodes are the corresponding log(h). And the red ones describe the calculation processes with our precondition, while the blue ones describe the corresponding processes without our precondition. The dashed lines approximate the slopes of the corresponding polylines.

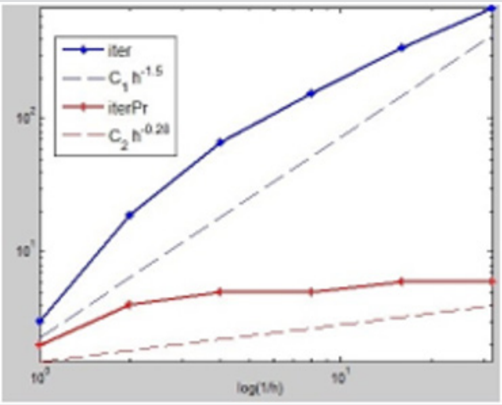

Figure 2: The figure describes the case of γ1 = 2h and γ2 = 100, where the ordinate values of these nodes on these polylines are iterative numbers which are the dates from Table 2, while the abscissa values of these nodes are the corresponding log(h). And the red ones describe the calculation processes with our precondition, while the blue ones describe the corresponding processes without our precondition. The dashed lines approximate the slopes of the corresponding polylines.

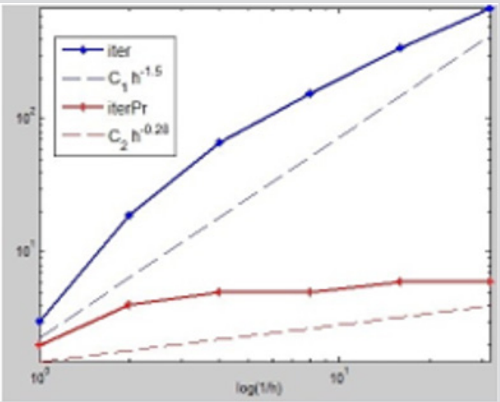

Figure 3: The figure describes the case of γ1 = h and γ2 = 50, where the ordinate values of these nodes on these polylines are iterative numbers which are the dates from Table 3, while the abscissa values of these nodes are the corresponding log(h). And the red ones describe the calculation processes with our precondition, while the blue ones describe the corresponding processes without our precondition. The dashed lines approximate the slopes of the corresponding polylines.

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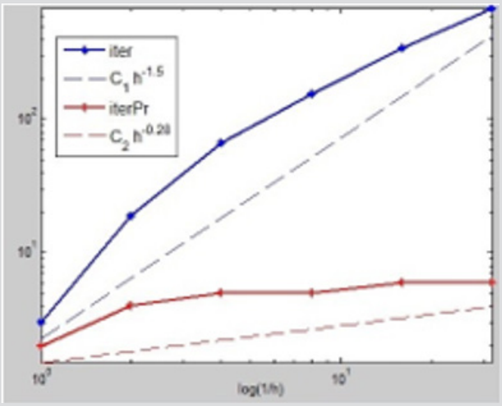

Figure 4: The figure describes the case of γ1 = 2h and γ2 = 50, where the ordinate values of these nodes on these polylines are iterative numbers which are the dates from Table 4, while the abscissa values of these nodes are the corresponding log(h). And the red ones describe the calculation processes with our precondition, while the blue ones describe the corresponding processes without our precondition. The dashed lines approximate the slopes of the corresponding polylines.

It is seen that as h decreases, the iteration numbers do not increase, which confirm our theoretical results reflect that our preconditioner is efficient and this preconditioned DD method is optimal.

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