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On One Aspect of Constrained Bayesian Method for Testing Directional Hypotheses

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Abstract

The paper discusses the application of constrained Bayesian method (CBM) of testing the directional hypotheses. It is proved that decision rule of CBM restricts the mixed directional false discovery rate (mdFDR) and total Type III error rate as well.

Keywords: Cbm; Hypotheses Testing; Mixed Directional False Discovery Rate; Type Iii Error Rate

Introduction

Directional hypotheses testing problem arises in many biomedical applications [1]. For parametrical models, the problem of testing directional hypotheses can be stated as $H_0: \theta = \theta_0$ vs. $H_-: \theta < \theta_0$ or $H_+: \theta > \theta_0$, where θ is the parameter of the model, θ_0 is known. A review of the works where the methods of testing the directional hypotheses are given can be found, for example, in [2,3]. Bayesian decision theoretical methodology for testing the directional hypotheses was developed and compared with the frequents method in [2]. While a new approach to the statistical hypotheses testing, called Constrained Bayesian Methods (CBM), was developed and applied alongside to other types of hypotheses to the directional hypotheses in [3]. One specific aspect of CBM for testing directional hypotheses, in particular, the fact that it restricts both the mixed directional false discovery rate (mdFDR) and total Type III error rate is proved below.

Constrained Bayesian Method and Error Rates

Let's denote: Γ_0 Γ_-, Γ_+ and are the regions of acceptance of the above-stated hypotheses H_0, H_- and H_+, respectively, and let us consider the following losses

$$L_1(H_i,H_j)=\begin{cases}0 & at \quad i=j,\\ L_1(H_i,H_j) & at \quad i\neq j,\end{cases}\quad L_2(H_i,H_j)=\begin{cases}0 & at \quad i\neq j,\\ L_2(H_i,H_i) & at \quad i=j,\end{cases} \qquad (1)$$

Where $L_1(H_i,H_j)$ and $L_2(H_i,H_j)$ are the losses of incorrectly accepted and incorrectly rejected hypotheses. It is clear that the " $0-1$" loss function is a private case of the step-wise loss (1). For loss functions (1), one of possible statement of CBM takes the form [3-5].

$$r_\delta = \min_{\{\Gamma_-,\Gamma_0,\Gamma_+\}}\Big\{p(H_-)\Big[L_1(H_-,H_0)\int_{\Gamma_0}p(x\,|\,H_-)dx + L_1(H_-,H_+)\int_{\Gamma_+}p(x\,|\,H_-)dx\Big]+$$
$$+\,p(H_0)\Big[L_1(H_0,H_-)\int_{\Gamma_-}p(x\,|\,H_0)dx + L_1(H_0,H_+)\int_{\Gamma_+}p(x\,|\,H_0)dx\Big]+$$
$$+\,p(H_+)\Big[L_1(H_+,H_-)\int_{\Gamma_-}p(x\,|\,H_+)dx + L_1(H_+,H_0)\int_{\Gamma_0}p(x\,|\,H_+)dx\Big]\Big\}, \qquad (2)$$

Subject to the averaged loss of incorrectly rejected hypotheses

$$p(H_-)L_2(H_-,H_-)\int_{\Gamma_-}p(x\,|\,H_-)dx + p(H_0)L_2(H_0,H_0)\int_{\Gamma_0}p(x\,|\,H_0)dx + p(H_+)L_2(H_+,H_+)\int_{\Gamma_+}p(x\,|\,H_+)dx \geq$$
$$\geq p(H_-)L_2(H_-,H_-)+p(H_0)L_2(H_0,H_0)+p(H_+)L_2(H_+,H_+)-r_1, \qquad (3)$$

Where $p(H_i)$ is the a priori probability of hypothesis H_i,- $p(x|H_i)$

Denotes the marginal density of x given H_i, i.e. $p(x\,|\,H_i)=\int_{\Theta_i}p(x\,|\,\theta)\pi(\theta\,|\,H_i)d\theta$, $\pi(\theta\,|\,H_i)$ is a priori density with support Θ_i, $i\in(-,0,+)$ and r_1 is some real number determining the level of the averaged loss of incorrectly rejected hypothesis.

$$L_1(H_i,H_j)=\begin{cases}0 & at \quad i=j,\\ K_1 & at \quad i\neq j,\end{cases},\quad L_2(H_i,H_j)=\begin{cases}0 & at \quad i\neq j,\\ K_0 & at \quad i=j.\end{cases} \qquad (4)$$

In this case, stated problem (2), (3) takes the following form

$$r_\delta = \min_{\{\Gamma_-,\Gamma_0,\Gamma_+\}}\Big\{p(H_-)\cdot K_1\cdot\Big[\int_{\Gamma_0}p(x\,|\,H_-)dx+\int_{\Gamma_+}p(x\,|\,H_-)dx\Big]+$$
$$+\,p(H_0)\cdot K_1\cdot\Big[\int_{\Gamma_-}p(x\,|\,H_0)dx+\int_{\Gamma_+}p(x\,|\,H_0)dx\Big]+$$
$$+\,p(H_+)\cdot K_1\cdot\Big[\int_{\Gamma_-}p(x\,|\,H_+)dx+\int_{\Gamma_0}p(x\,|\,H_+)dx\Big]\Big\}, \qquad (5)$$

Subject to

$$p(H_-)\int_{\Gamma_-}p(x\,|\,H_-)dx+p(H_0)\int_{\Gamma_0}p(x\,|\,H_0)dx+p(H_+)\int_{\Gamma_+}p(x\,|\,H_+)dx\geq 1-\frac{r_1}{K_0}. \qquad (6)$$

By solving constrained optimization problem (5), (6), using the Lagrange multiplier method, we obtain

$$\Gamma_- = \left\{ x : K_1 \cdot \left(p(H_0) p(x \mid H_0) + p(H_+) p(x \mid H_+) \right) < \lambda \cdot K_0 \cdot p(H_-) p(x \mid H_-) \right\},$$

$$\Gamma_0 = \left\{ x : K_1 \cdot \left(p(H_-) p(x \mid H_-) + p(H_+) p(x \mid H_+) \right) < \lambda \cdot K_0 \cdot p(H_0) p(x \mid H_0) \right\}, \quad (7)$$

$$\Gamma_+ = \left\{ x : K_1 \cdot \left(p(H_-) p(x \mid H_-) + p(H_0) p(x \mid H_0) \right) < \lambda \cdot K_0 \cdot p(H_+) p(x \mid H_+) \right\},$$

Where Lagrange multiplier λ is determined so that in (6) equality takes place. For optimality of testing directional hypotheses, different concepts such as: mixed directional false discovery rate *(mdFDR),* directional false discovery rate *(DFDR)* and the Type III errors are offered in [1,6-10]. Let us consider *mdFDR* which is the exected proportion of falsely selecting H_- or H_+. In our case, it has the following form

$$mdFDR = P(x \in \Gamma_- \mid H_0) + P(x \in \Gamma_- \mid H_+) + P(x \in \Gamma_+ \mid H_0) + P(x \in \Gamma_+ \mid H_-) =$$

$$= \int_{\Gamma_-} p(x \mid H_0) dx + \int_{\Gamma_-} p(x \mid H_+) dx + \int_{\Gamma_+} p(x \mid H_0) dx + \int_{\Gamma_+} p(x \mid H_-) dx. \quad (8)$$

According to [6,9]. The Type III error rate is

$$Type-III \quad error \quad rate = P(x \in \Gamma_- \mid H_0) + P(x \in \Gamma_+ \mid H_0). \quad (9)$$

But in [10]. It is defined as

$$Type-III \quad error \quad rate = P(x \in \Gamma_- \mid H_+) + P(x \in \Gamma_+ \mid H_-). \quad (10)$$

Let us denote the Type III error rate (9) as ERR_{III}^{T} and Type III error rate (4) as ERR_{III}^{K}.

From (8), (9) and (10), it follows that

$$mdFDR = ERR_{III}^{T} + ERR_{I}. \quad (11)$$

Theorem

When satisfying the condition $\frac{1}{p_{\min}} \cdot \frac{r_1}{K_0} = \alpha$, where $p_{\min} = \min\{p(H_-), p(H_0), p(H_+)\}$, CBM with restriction level of (6) ensures a decision rule with mixed directional false discovery rate or with total Type III error rate less or equal to , *i.e.* with $mdFDR = ERR_{III}^{T} + ERR_{III}^{K} \le \alpha$.

Proof: Because of specificity of decision rules of CBM, alongside of hypotheses acceptance regions, the regions of impossibility of making a decision exist. Therefore, instead of the condition

$$\int_{\Gamma_-} p(x \mid H_i) dx + \int_{\Gamma_0} p(x \mid H_i) dx + \int_{\Gamma_+} p(x \mid H_i) dx = 1, \quad i \in (-, 0, +),$$

Of classical decision rules, the following condition is fulfilled in CBM

$$\int_{\Gamma_-} p(x \mid H_i) dx + \int_{\Gamma_0} p(x \mid H_i) dx + \int_{\Gamma_+} p(x \mid H_i) dx + P(imd \mid H_i) = 1, \quad i \in (-, 0, +), \quad (12)$$

Where $P(imd \mid H_i)$ is the probability of impossibility of making a decision [3,11,12].

Taking into account (12), condition (6) can be rewritten as follows

$$p(H_-) \int_{\Gamma_-} p(x \mid H_-) dx + p(H_0) \int_{\Gamma_0} p(x \mid H_0) dx + p(H_+) \int_{\Gamma_+} p(x \mid H_+) dx =$$

$$= p(H_-) \left[1 - \int_{\Gamma_+} p(x \mid H_-) dx - \int_{\Gamma_0} p(x \mid H_-) dx - P(imd \mid H_-) \right] +$$

$$+ p(H_0) \left[1 - \int_{\Gamma_-} p(x \mid H_0) dx - \int_{\Gamma_+} p(x \mid H_0) dx - P(imd \mid H_0) \right] +$$

$$+ p(H_+) \left[1 - \int_{\Gamma_-} p(x \mid H_+) dx - \int_{\Gamma_0} p(x \mid H_+) dx - P(imd \mid H_+) \right] =$$

$$= 1 - p(H_-) \left[\int_{\Gamma_+} p(x \mid H_-) dx + \int_{\Gamma_0} p(x \mid H_-) dx + P(imd \mid H_-) \right] -$$

$$- p(H_0) \left[\int_{\Gamma_-} p(x \mid H_0) dx + \int_{\Gamma_+} p(x \mid H_0) dx + P(imd \mid H_0) \right] -$$

$$- p(H_+) \left[\int_{\Gamma_-} p(x \mid H_+) dx + \int_{\Gamma_0} p(x \mid H_+) dx + P(imd \mid H_+) \right] \ge 1 - \frac{r_1}{K_0}. \quad (13)$$

Let us denote $p_{\min} = \min\{p(H_-), p(H_0), p(H_+)\}$. Then, from (13), we have

$$\int_{\Gamma_+} p(x \mid H_-) dx + \int_{\Gamma_0} p(x \mid H_-) dx + P(imd \mid H_-) +$$

$$\int_{\Gamma_-} p(x \mid H_0) dx + \int_{\Gamma_+} p(x \mid H_0) dx + P(imd \mid H_0) +$$

$$\int_{\Gamma_-} p(x \mid H_+) dx + \int_{\Gamma_0} p(x \mid H_+) dx + P(imd \mid H_+) \le \frac{1}{P_{\min}} \cdot \frac{r_1}{K_0}. \quad (14)$$

Taking into account (8), we write

$$mdFDR + \int_{\Gamma_0} p(x \mid H_-) dx + \int_{\Gamma_0} p(x \mid H_+) dx + P(imd \mid H_-) +$$

$$+ P(imd \mid H_0) + P(imd \mid H_+) \le \frac{1}{P_{\min}} \cdot \frac{r_1}{K_0},$$

This proves the statement of the theorem.

Conclusion

One more property of the optimality of CBM when testing the directional hypotheses is shown. In particular, when testing the directional hypotheses, the optimal decision rule of CBM restricts both the mixed directional false discovery rate (mdFDR) and total Type III error rate.

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