Introduction

Directional hypotheses testing problem arises in many biomedical applications [1]. For parametrical models, the problem of testing directional hypotheses can be stated as \( H_0 : \theta = \theta_0 \) vs. \( H_1 : \theta < \theta_0 \) or \( H_1 : \theta > \theta_0 \), where \( \theta \) is the parameter of the model, \( \theta_0 \) is known. A review of the works where the methods of testing the directional hypotheses are given can be found, for example, in [2,3]. Bayesian decision theoretical methodology for testing the directional hypotheses was developed and compared with the frequentist method in [2]. While a new approach to the statistical hypotheses testing, called Constrained Bayesian Methods (CBM), was developed and applied alongside to other types of hypotheses to the directional hypotheses in [3]. One specific aspect of CBM for testing directional hypotheses, in particular, the fact that it restricts both the mixed directional false discovery rate (mdFDR) and total Type III error rate as well.

Constrained Bayesian Method and Error Rates

Let’s denote: \( \alpha_+ \), \( \alpha_0 \), \( \alpha_- \) are the regions of acceptance of the above-stated hypotheses \( H_0 \), \( H_+ \) and \( H_- \), respectively, and let us consider the following losses

\[
L_+(H,H_0) = \begin{cases} 0 & \text{at } i = j, \\ L(H,H_0) & \text{at } i \neq j, \\ L(H,H_0) & \text{at } i = j, \end{cases}
\]

\[
L_-(H,H_0) = \begin{cases} 0 & \text{at } i = j, \\ L(H,H_0) & \text{at } i \neq j, \\ L(H,H_0) & \text{at } i = j, \end{cases}
\]

Where \( L(H,H_0) \) and \( L(H,H_+) \) are the losses of incorrectly accepted and incorrectly rejected hypotheses. It is clear that the “\( 0^-1^+ \)” loss function is a private case of the step-wise loss (1). For loss functions (1), one of possible statement of CBM takes the form

\[
r_+ = \min_{\theta \in \theta_+} \left\{ p(H_0) [\int L(H_0,H_0) p(x|H_0) dx + L(H_0,H_+) p(x|H_+)] + \right.
\]

\[
L(H_0,H_+) \left[ \int p(x|H_+)^2 dx + \int p(x|H_-) dx \right] +
\]

\[
\left. + \right. \left( p(H) - p(H_0) \right) \left[ \int L(H,H_0) p(x|H_0) dx + \int L(H,H_+) p(x|H_+) dx \right].
\]

Subject to the averaged loss of incorrectly rejected hypotheses

\[
p(H)[L(H,H_0) \int p(x|H_0) dx + L(H,H_+) \int p(x|H_+) dx + L(H,H_-) \int p(x|H_-) dx] \\
\geq p(H)[L(H,H_0) + L(H,H_+) + L(H,H_-) - r_0],
\]

Where \( p(H) \) is the a priori probability of hypothesis \( H \), \( r_0 \) is some real number determining the level of the averaged loss of incorrectly rejected hypothesis.

\[
L_i(H,H_i = \begin{cases} 0 & \text{at } i = j, \\ K_i & \text{at } i \neq j, \end{cases}
\]

In this case, stated problem (2), (3) takes the following form

\[
r_+ = \min_{\theta \in \theta_+} \left\{ p(H) [\int L(H,H_0) p(x|H_0) dx + \int L(H,H_+) p(x|H_+) dx] + \right.
\]

\[
L(H_0,H_+) \left[ \int p(x|H_0)^2 dx + \int p(x|H_-) dx \right] +
\]

\[
\left. + \right. \left( p(H) - p(H_0) \right) \left[ \int L(H,H_0) p(x|H_0) dx + \int L(H,H_+) p(x|H_+) dx \right].
\]

Subject to

\[
p(H)[\int p(x|H_0) dx + \int p(x|H_+) dx + \int p(x|H_-) dx + \int p(x|H_-) dx] \\
\geq p(H)[L(H,H_0) + L(H,H_+) + L(H,H_-) - r_0].
\]

By solving constrained optimization problem (5), (6), using the Lagrange multiplier method, we obtain

\[
r_+ = \frac{\int p(x|H_0)^2 dx + \int p(x|H_-) dx}{K_+}.
\]
\[ \Gamma = \{ x : K_i \{ p(H_0)p(x | H_0) + p(H_1)p(x | H_1) \} < \lambda \cdot K_i \{ p(H_1)p(x | H_1) \} \}, \]
\[ \Gamma_i = \{ x : K_i \{ p(H_1)p(x | H_1) + p(H_0)p(x | H_0) \} < \lambda \cdot K_i \{ p(H_0)p(x | H_0) \} \}. \]  

(7)  

Where \( \lambda \) is determined so that in (6) equality takes place. For optimality of testing directional hypotheses, different concepts such as: mixed directional false discovery rate (mdFDR), directional false discovery rate (DFDR) and the Type III errors are offered in [1,6-10]. Let us consider mdFDR which is the expected proportion of falsely selecting \( H \) or \( H_i \). In our case, it has the following form

\[ \text{mdFDR} = \int_{\Gamma} p(x | \Gamma \cap H_0) + \int_{\Gamma} p(x | \Gamma \cap H_1) + \int_{\Gamma} p(x | \Gamma \cap H_0) + \int_{\Gamma} p(x | \Gamma \cap H_1) = \]
\[ = \int_{\Gamma} p(x | H_0)dx + \int_{\Gamma} p(x | H_1)dx + \int_{\Gamma} p(x | H_0)dx + \int_{\Gamma} p(x | H_1)dx. \]  

(8)

According to [6,9], the Type III error rate is

\[ \text{Type - III error rate} = P(x \in \Gamma \cap H_0) + P(x \in \Gamma \cap H_1). \]  

(9)

But in [10], it is defined as

\[ \text{Type - III error rate} = P(x \in \Gamma \cap H_0) + P(x \in \Gamma \cap H_1). \]  

(10)

Let us denote the Type III error rate (9) as \( \text{ERR}_3 \) and Type III error rate (4) as \( \text{ERR}_T \).

From (8), (9) and (10), it follows that

\[ \text{mdFDR} = \text{ERR}_3 + \text{ERR}_T. \]  

(11)

Theorem:

When satisfying the condition \( \frac{1}{\lambda} = \frac{K_i}{K} \), where \( \lambda = \min \{ \lambda | \xi |, \xi | \eta |, \eta | \eta \} \), CBM with restriction level of (6) ensures a decision rule with mixed directional false discovery rate or with total Type III error rate less or equal to \( \lambda \), i.e. with \( \text{mdFDR} = \text{ERR}_3 + \text{ERR}_T \leq \lambda \).

Proof: Because of specificity of decision rules of CBM, alongside of hypotheses acceptance regions, the regions of impossibility of making a decision exist. Therefore, instead of the condition

\[ \int_{\Gamma} p(x | H_0)dx + \int_{\Gamma} p(x | H_1)dx + \int_{\Gamma} p(x | H_0)dx = 1, \quad i \in \{ H_0, H_1 \}, \]  

(12)

Of classical decision rules, the following condition is fulfilled in CBM

\[ \int_{\Gamma} p(x | H_0)dx + \int_{\Gamma} p(x | H_1)dx + \int_{\Gamma} p(x | H_0)dx + P(\text{mdFDR} | H_0) = 1, \quad i \in \{ H_0, H_1 \}, \]  

(13)

Where \( P(\text{mdFDR} | H) \) is the probability of impossibility of making a decision [3,11,12].

Taking into account (12), condition (6) can be rewritten as follows

\[ P(H_1) \int_{\Gamma} p(x | H_1)dx + p(H_0) \int_{\Gamma} p(x | H_0)dx + p(H_1) \int_{\Gamma} p(x | H_1)dx = \]
\[ = P(H_1) \left[ \int_{\Gamma} p(x | H_1)dx - \int_{\Gamma} p(x | H_0)dx + P(\text{mdFDR} | H_1) \right] + \]
\[ + P(H_0) \left[ \int_{\Gamma} p(x | H_0)dx - \int_{\Gamma} p(x | H_1)dx - P(\text{mdFDR} | H_0) \right] + \]
\[ + P(H_i) \left[ \int_{\Gamma} p(x | H_i)dx - \int_{\Gamma} p(x | H_j)dx - P(\text{mdFDR} | H_i) \right] = \]
\[ = 1 - P(H_1) \int_{\Gamma} p(x | H_1)dx + \int_{\Gamma} p(x | H_1)dx + P(\text{mdFDR} | H_1) - \]
\[ - P(H_0) \int_{\Gamma} p(x | H_0)dx + \int_{\Gamma} p(x | H_0)dx + P(\text{mdFDR} | H_0) - \]
\[ - P(H_i) \int_{\Gamma} p(x | H_i)dx + \int_{\Gamma} p(x | H_i)dx + P(\text{mdFDR} | H_i) \].

(13)

Let us denote \( P_{\min} = \min \{ P(H_1), P(H_0), P(H_i) \} \). Then from (13), we have

\[ \int_{\Gamma} p(x | H_1)dx + \int_{\Gamma} p(x | H_0)dx + P(\text{mdFDR} | H_1) + \]
\[ + \int_{\Gamma} p(x | H_0)dx + \int_{\Gamma} p(x | H_1)dx + P(\text{mdFDR} | H_0) + \]
\[ + \int_{\Gamma} p(x | H_i)dx + \int_{\Gamma} p(x | H_i)dx + P(\text{mdFDR} | H_i) \leq \frac{1}{P_{\min}} \frac{\lambda}{K}. \]  

(14)

Taking into account (8), we write

\[ \text{mdFDR} + \int_{\Gamma} p(x | H_1)dx + \int_{\Gamma} p(x | H_0)dx + P(\text{mdFDR} | H_1) + \]
\[ + \int_{\Gamma} p(x | H_0)dx + \int_{\Gamma} p(x | H_1)dx + P(\text{mdFDR} | H_0) + \]
\[ + \int_{\Gamma} p(x | H_i)dx + \int_{\Gamma} p(x | H_i)dx + P(\text{mdFDR} | H_i) \leq \frac{1}{P_{\min}} \frac{\lambda}{K}. \]  

This proves the statement of the theorem.

Conclusion:

One more property of the optimality of CBM when testing the directional hypotheses is shown. In particular, when testing the directional hypotheses, the optimal decision rule of CBM restricts both the mixed directional false discovery rate (mdFDR) and total Type III error rate.

References:


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